Kissane, B. (1997) The graphics calculator: Imagine the probabilities. In: Clarke, D., Clarkson, P., Gronn, D., Horne, M., Lowe, L., Mackinlay, M. and McDonough, A., (eds.) Mathematics: Imagine the possibilities. Mathematical Association of Victoria, Brunswick, VIC., pp. 487-497.

http://researchrepository.murdoch.edu.au/6373

Using a graphics calculator: Imagine the probabilities*

Barry Kissane, School of Education, Murdoch University, WA

Introduction

The graphics calculator is the first instance of technology since the invention of the scientific calculator which might be expected to be accessible to all the students in a secondary school mathematics classroom all of the time. This accessibility is critical for mathematics education. (Kissane, 1995a). The main purpose of this paper is to suggest some of the ways in which teaching and learning probability might be positively affected by the ready availability of a graphics calculator.

Although graphics calculators have been used in secondary schools in affluent countries now for more than a decade, they are still frequently regarded as 'new' forms of technology. But this is not the only part of the mythology of graphics calculators. (Kissane, 1997a). Newcomers to this technology, or those reluctant to use it, frequently think that its main educational uses involve drawing and exploring graphs of functions. In those settings in which external examinations are dominant, the myth that graphics calculators are mainly useful for dealing with examination questions is not hard to find. These are rather limited views of the significance of the graphics calculator for secondary mathematics, however. If the graphics calculator is regarded as the first example of a genuinely personal technology, students should be able to use it for many more activities than exploring graphs of functions. Even a casual perusal of recent publications concerned with the use of graphics calculators (such as Kissane, 1997c, 1997d) will make it clear that *most* of the significant educational uses of graphics calculators involve activities other than drawing graphs of functions. Nor should we be restricted to the limiting metaphor of the calculator as a 'tool', or the refinement to a 'tool for examinations'; the metaphor of the calculator as a laboratory, facilitating exploration, is of more relevance to this paper. (Kissane, 1995b).

An earlier paper (Kissane, 1997b) provided an overview of the different kinds of ways in which the probability curriculum might be affected by the graphics calculator. In the present paper, attention is restricted to the use of the calculator to simulate events, and thus to help to develop insight into the nature of chance processes. The formal study of probability, late in secondary school, should rest upon appropriate experiences and insights, which the graphics calculator can help to provide.

Basic Calculator Capabilities

The significance of the graphics calculator for probability is a direct consequence of four capabilities of all modern graphics calculators. These are (i) random number generation; (ii) mathematical functions; (iii) data analysis; and (iv) programmability. In this section, these capabilities are described and sample screens are given for the Casio cfx-9850G calculator.

Random Number Generation

Most scientific calculators and all graphics calculators have a command for generating (pseudo-) random numbers uniformly on the interval (0,1). An advantage of graphics calculators over scientific calculators is the ease of generating *many* random numbers. The left screen of Figure 1 shows that the relevant command on the Casio cfx-9850G is *Ran#*, and that a succession of numbers is obtained with repeated tapping of the *Execute* key.

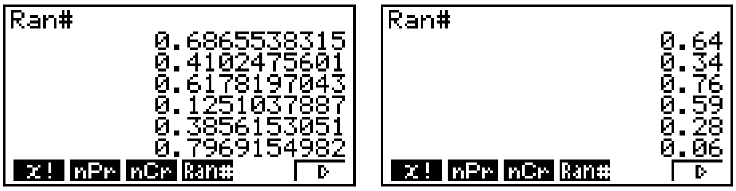

Figure 1. Generation of successive random numbers on (0,1)

As for digits in published tables of random numbers, it is possible to regard these random numbers as providing a succession of random digits, uniformly distributed on the set {0,1,2,3,4,5,6,7,8,9}, provided the initial zeroes and the decimal point are ignored. Alternatively, the right screen of Figure 1 shows that the calculator can be instructed to generate random numbers truncated to a particular number of decimal places, by adjusting the screen display parameters. In this case, setting the calculator to give all results to two decimal places and then ignoring the decimal point on the screen provides ready access to 2-digit random numbers, uniformly distributed from 00 to 99, a bit more conveniently than reading successive pairs of digits as a two-digit number.

Repeatedly tapping an *Execute* or *Enter* key can quickly become tedious, and early random numbers are lost from view once the calculator screen becomes full. A more efficient way of generating many random numbers is to use the tabulation facility of a calculator, after defining a function with the random number command. Figure 2 shows an example of this, for each of two functions. Notice that the successive random numbers in each column (i.e. each of the two functions) are different, although their definitions are identical.

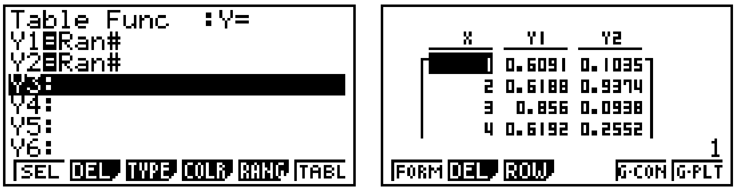

Figure 2. Using tables to generate many random numbers on (0,1)

A third, and particularly useful, way of generating random numbers is to use the list capabilities of a calculator to generate a finite sequence of numbers. If these are stored directly into a list, they can be analysed using the data analysis capabilities of the calculator. Figure 3 shows an example of this kind of random number generation, using the *Seq* command to generate a set of five random numbers.

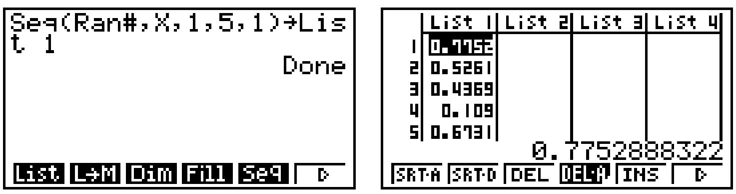

Figure 3. Using a sequence command to generate a set of five random numbers on (0,1)

Mathematical Functions

By themselves, random numbers on (0,1) are cumbersome to work with, and the capability of graphics calculators to transform them to a more suitable form is especially important. Linear transformations can be constructed on a calculator to generate random variables to suit particular contexts. Many of these use an integer command, *Int*, which gives the integer part of a number. For example, the calculator command *Int(Ran# + 0.2)* generates the number 1 for 20% of the time and the number 0 for 80 % of the time; that is, it simulates a Bernoulli random variable with probability of success $p = 0.2$. Similarly, the command *Int(6Ran#+1)* generates an integer from the set {1,2,3,4,5,6} with a uniform distribution, thus simulating a roll of a single fair die. Commands like these might be used to simulate successive values, a table of values or a list of values for a context of interest. For example, Figure 4 shows how the die command can be used to generate a table of values, which can then be scrolled.

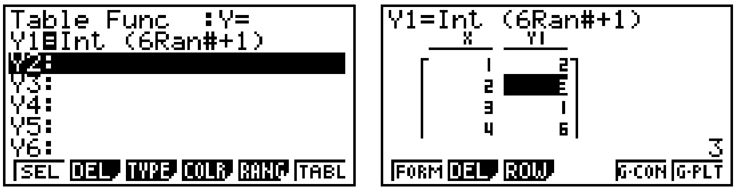

Figure 4. Simulating rolling a die to produce a scrollable list

As well as using a transformation to generate a random variable, the calculator can be used to generate random variables comprising other elemental random variables. For example, the screens in Figure 5 show how the calculator can be used to simulate tossing a pair of dice, and adding their totals (shown as Y3).

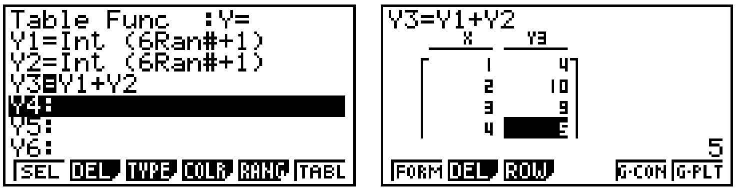

Figure 5. Simulating rolling a pair of standard dice

In a similar way, the basic building block of a Bernoulli random variable can be used to construct a binomial random variable (the sum of several independent and identically distributed Bernoulli random variables). Figure 6 shows two ways of doing this. After defining the appropriate function (Y2) in the left screen, the middle screen shows that single observations can be generated by recalling function Y2 and then pressing the *Execute* key in the home screen. The third screen shows how simulated observations can be stored into a table for ease of analysis.

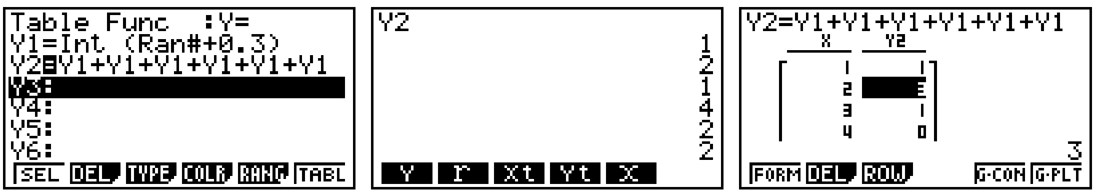

Figure 6. Simulating observations for a binomial random variable with $n = 6$ and $p = 0.3$

Data Analysis

Although learning some aspects of probability can be enhanced by informal observation of simulated data, usually a substantial body of data is involved and more efficient mechanisms for making sense of the data are needed. All graphics calculators include data analysis capabilities, which can be used to summarise and determine trends in simulated data. Figure 7 shows an example. After the calculator was used to simulate the tossing of a standard pair of dice 250 times, the resulting data were transferred to a list and thus available for analysis in statistics mode. For these data, a histogram is a suitable graphical display, while an assortment of statistics are also available, some of which are shown in the screen at the right of Figure 7.

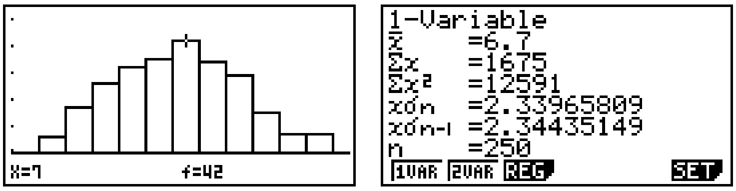

Figure 7. Analysing graphically and numerically 250 simulated tosses of a pair of standard dice

The mechanism for transferring simulated data into the data analysis area of the calculator and the associated memory limitations vary from calculator to calculator. It is, of course, unrealistic to transfer large amounts of data manually. Both the Casio cfx-9850G and the smaller Casio fx-7400G include a *List Memory* command for automatic transfer of data from a table to a list, which is particularly useful for this purpose. Other models of calculators are more awkward to use in this respect, since a (complicated) sequence command must be used instead of a table. Both the Casio calculators will accommodate a list of up to 255 elements, subject to the available calculator memory. Other calculators have different constraints; for example, while a Texas Instruments TI-83 will accommodate lists of up to 999 elements, both a TI-82 and a TI-80 will only permit lists of up to 99 elements, again subject to the available calculator memory.

Programmability

All graphics calculators can be programmed, so that they can be tailored to conduct particular tasks. While writing programs requires some knowledge of a calculator's programming language, many simple programs can be written by beginners who know how to operate the calculator. More sophisticated programs require more expertise to write, but not to *use*. Even short programs can be used by people unfamiliar with programming, once the program has been entered into their calculator, either manually or electronically. (Kissane, 1994).

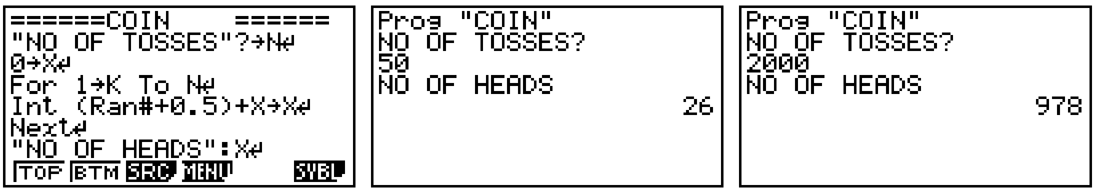

Figure 8. Using a short program to simulate tossing a fair coin

A simple example is shown in Figure 8. The program in the left screen simulates tossing a fair coin a number of times (*N*) determined by the user. The number of heads (i.e a score of 1) is counted as variable *X* and reported. The middle screen shows a result of using the program with $N = 50$, while the right screen shows a simulation of 2000 tosses. Although the calculator is fairly slow to simulate 2000 coin tosses (taking around 40 seconds), it is a much more efficient (and quieter!) data collection method than manual tossing of a coin 2000 times.

Classroom Experimentation

The curriculum in probability, particularly informal probability, can be significantly enhanced by engagement in activities involving chance. These in turn can be devised by making use of the calculator capabilities described in the previous section. The *National Statement* drew attention to this sort of possibility some years ago:

> Such activities also provide the basis for more formal study in later school years. Students should investigate chance events, estimate probabilities experimentally (that is, empirically) and determine them analytically (that is, theoretically). Many situations which are difficult to deal with analytically and which cannot be directly experimented with are readily accessible through simulation methods. Students can solve quite complex problems using simple simulation techniques, particularly if appropriate computing facilities are available. (Australian Education Council, 1990, p.163)

The question of availability of computing facilities is much easier to resolve when students have access to graphics calculators than by trying to provide significant access to larger kinds of microcomputers, as already noted above. (Kissane, 1995a, 1995b).

There is not space in this brief paper to elaborate on the many kinds of uses graphics calculators might have for the probability curriculum. Instead, some brief examples of four different kinds are presented here; many more can be generated in these categories by teachers familiar with both the probability curriculum and with graphics calculators.

Informal Experiences

An important reason for practical experimentation in the early study of chance is to build intuitions and insights upon which more formal study of probability might later rest. A student using a calculator to generate and study random data may be less likely to develop standard misconceptions.

For example, while students may accept at one level that a standard die is 'equally likely' to yield each of its six possible results, they may nonetheless expect that this means that one sixth of the tosses ought to fall on each possibility every time. Rolling a die a number of times and recording the outcomes will help such a student get a better feel for the realities of randomness. Using a calculator to help with the data collection and display can contribute to this process in an efficient way. Once a student has learned how to simulate a die roll, using *Int(6Ran# + 1)*, and knows their way around their calculator, they can produce a succession of histograms such as those in Figure 9. At the very least, these kinds of data will help students see that something different is likely to happen each time, and that each of the numbers does not occur precisely 20 times each. Such experiences would seem important to developing an understanding later of probability as 'long run relative frequency'.

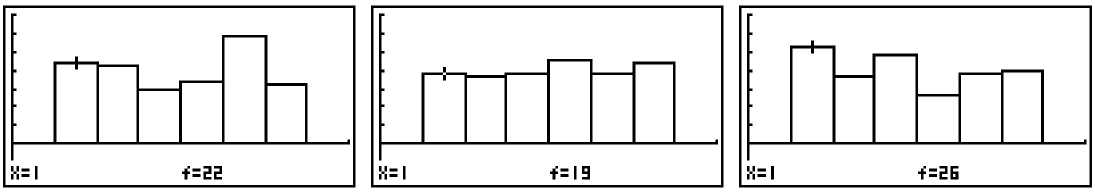

Figure 9. Three histograms resulting from simulating 120 die rolls three times

As a second example, studying simulated events with very high probabilities may help dispel the misconception that something with a high probability must always take a long time to fail. The command *Int(Ran# + 0.95)* will simulate a success (ie., a score of 1) 95% of the time and a failure (a score of 0) 5% of the time. However, contrary to popular misconception, the sequence of successes and failures will not consist of about one failure in every twenty. Indeed, as Figure 10 shows for two successive experiments of this kind, sometimes the unlikely happens quite early in a sequence, although in the long run, it may still occur only about 5% of the time.

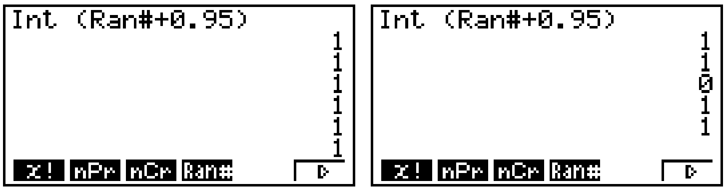

Figure 10. Simulating a sequence of very likely events

The practical consequences of this sort of understanding of elementary probability may be considerable, depending on the context. Consider events such as the success of a medical operation, of a form of contraception, or of a bus arriving at a destination on time.

Taking Advantage of Class Sizes

For most educational purposes, the disadvantages of large class sizes outweigh any advantages, which are mainly social in nature. But when chance activities are concerned, a larger class provides more data from which to understand the phenomenon in question than does a smaller class.

To give an example, consider again the program "Coin" illustrated in Figure 8. If each student in a class uses the program, they will see that they do not get exactly the same result each time when they use the program a few times in succession. However, a powerful use of the numbers of students in the class is to pool everyone's results (perhaps by constructing a stem and leaf diagram on the blackboard) and see what can be learned from them. The results of one such experiment are shown in Figure 11, the left screen of which shows the results of thirty different executions of the program.

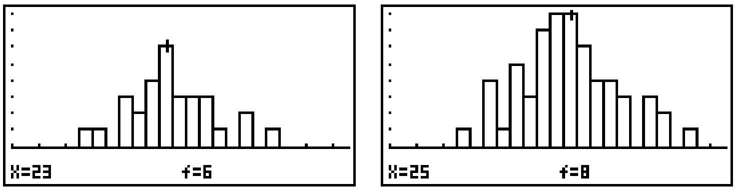

Figure 11. Graphs of 30 and 60 separate simulations of the numbers of heads in 50 coin tosses

The right screen of Figure 11 shows a further sixty executions of the same program, which might easily be obtained by each student in a class running the program twice. Aggregations like these illustrate well an important aspect of chance events: that while individual events are quite unpredictable, there is unmistakable evidence of a consistency, and thus predicability, in larger amounts of data.

The same ideas are relevant to many probabilistic situations, and might well be taken advantage of in an informal way. To give another example, consider the very likely events described earlier and shown in Figure 10. If each student in a class performs the same simulation, a simple show of hands among the class can quickly give a sense of how long we are 'likely' to wait until there is a failure. With a class of 30 students, this kind of work provides access to thirty times the experience available to each student.

Monte Carlo Procedures

An important recent use of probability involves the solution of problems using Monte Carlo procedures, named after the small European country famous for its casino. The use of probability to solve problems not obviously involving ideas of chance was not possible until the development of technologies capable of generating a lot of random data and analysing the results. A good example is the determination of the area under the parabola $y = x^2$ between $x = 0$ and $x = 1$, *without* using the calculus. This problem was successfully solved by

Archimedes a very long time ago, but can also be tackled by a modern Monte Carlo procedure.

One approach is to generate a pair of random numbers, *A* and *B*, each on the interval (0,1) so that the point (*A*,*B*) is on the unit square shown in Figure 12. By comparing A^2 with *B*, the location of the point can be determined: it will be either under or over the curve. The proportion of a large number of random points that falls under the curve will provide a good estimate of the proportion of the area of the unit square that is under the curve. An efficient way of doing this is to use a small program, a condensed version of which is shown in the middle screen of Figure 12. In this case, the program (named *Archimed*, in honour of the great mathematician) provides a reasonable approximation to the correct result, using as few as 1000 points.

Figure 12. A Monte Carlo determination of the area under the curve $y = x^2$

As a second example, consider the problem of collecting cards in breakfast cereal boxes. If there is only one card in each box, there are six different cards, and the cards are distributed randomly to the boxes, how many boxes are you likely to need to purchase to get a full set of cards? This problem, although relatively easy to understand, is difficult to solve analytically; however a good sense of the answer can be obtained by a Monte Carlo procedure.

An easy solution involves representing each card as a score on a standard die, simulating successive die rolls as in Figure 4, and observing them carefully until each of a 1, 2, 3, 4, 5 and 6 have appeared. A class of students can do this quite quickly, although they may need advice to record each of the six digits as they occur to avoid miscounting. As noted in the previous section, a good sense of the range of likely results can be obtained by a simple process of pooling a class's results. It is also possible to write a small program to perform many successive experiments of this kind, of course; an advantage of writing a program is that it might be flexibly able to investigate what happens if the numbers of cards to be collected is changed.

These sorts of procedures might also be used to develop a feel for probability problems that are beyond the capacity of students to deal with, such as the Birthday Problem of deciding how likely it is that at least one pair of a randomly selected group of people of a certain size shares a birthday. A skilful user of the calculator can generate random 'birthdays' in the set {1,2,3,...,365} using the command shown in Figure 13. For example, a table of 30 such birthdays can be inspected to determine whether or not there are any matches. It can be difficult to actually find any matches, however; an efficient way of doing so is firstly to transfer the table to the data analysis part of the calculator and then sort the data, as shown in the final screen of Figure 13.

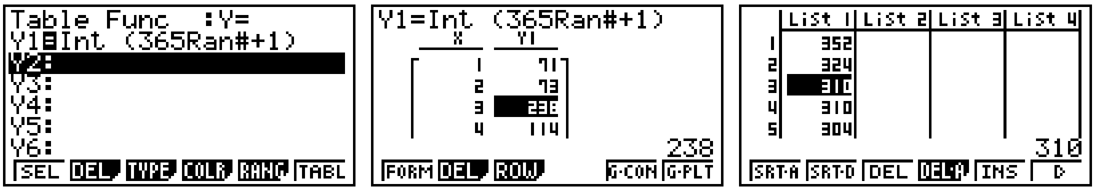

Figure 13. Simulating data for the Birthday Problem

If each of the students in a class carries out a simulation of this kind, and the proportion of matches found studied, a good intuitive feel for the surprising result can be obtained quite quickly. For more senior students, this may motivate an analytical treatment of the problem.

Understanding Sampling

A major use of probability, and the major link between probability and statistics, involves the understanding of what happens when samples are taken from a population. Exploring this link is an important part of the curriculum for the senior school, as suggested in the discussion about Band D in the *National Statement*:

> All students should develop at least an informal understanding of confidence intervals, even if a more technical treatment of this is inappropriate. Confidence intervals can also be used as an informal approach to hypothesis testing.
> (Australian Education Council, 1990, p.181)

Part of this exploration should involve the idea of a sampling distribution, and the graphics calculator can be used to good effect to construct empirical distributions of samples from a population.

For such purposes, it is essential to add a sampling functionality to the calculator, since no graphics calculator at present contains a command for this purpose. A small program can be written to take a random sample from a data set, regarded as a population, and store the sample in the calculator for analysis. (Although only a few lines long, there is not space here to provide a detailed listing of such a program.) Then, if each of the students in a class draws a random sample from the same data set, the raw material for an empirical sampling distribution is readily available. For example, each student might be asked to find and report on the mean and standard deviation of their sample, and the set of means and the set of standard deviations can be informally compared with the mean and standard deviation of the original population. A useful class exercise is to examine the effect of choosing samples of different sizes, so that the stability of larger samples will become evident. From such class work, students can begin to get an intuitive feel for the ideas behind the concept of a confidence interval by addressing the question of how much information about the population in question they expect a *single* sample to reveal.

Slightly more sophisticated programs might use the same idea to construct empirical sampling distributions directly. For example, the data shown in Figure 14 were generated in this way, by selecting samples of sizes 10, 20 and 40 respectively and then calculating the means of each sample. So each histogram, drawn on the same scale, shows the means of 50 random samples of a certain size.

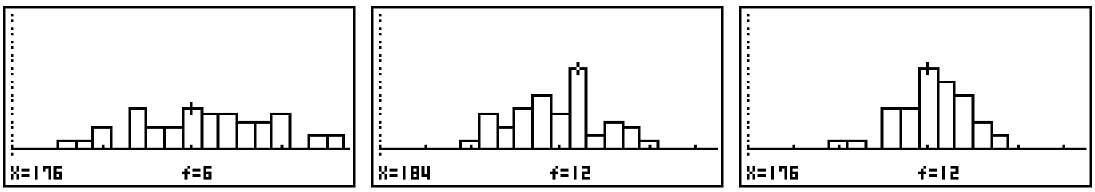

Figure 14. Distributions of the means of 50 samples each of sizes 10, 20 and 40 respectively

The three histograms show reasonably well that the sample means tend to be more tightly clustered as the sample size increases, and suggest that a symmetrical distribution such as a normal distribution may be a useful model for them. The three empirical sampling distributions all appear to be centred around similar values. Of course, the data analysis capabilities of the calculator can be used to quantify these sorts of observations. In these particular cases, the means were 187, 178 and 182 respectively while the standard deviations were 29.4, 19.1 and 14.6 respectively. The original population has a mean of 178 and a standard deviation of 94, so the theoretical expectation that the standard deviation of the sampling distribution of means of samples of size n is $\sigma/\sqrt{n}$ seems to hold, at least approximately.

Conclusion

The personal technology of the graphics calculator opens up many opportunities for informal exploration of concepts associated with probability. In this paper, examples have been chosen across a range of levels of sophistication and contexts, in the hope that they will be both illustrative of the considerable educational potential involved and suggestive of other examples. In most cases, all that is required of students and their teachers is sufficient familiarity with the workings of the calculator to deal smoothly with a few basic kinds of commands.

The graphics calculator cannot deal with all the needs of students coming to terms with the important concepts of probability, and should not be regarded as a replacement for other kinds of practical activities. However, it is an extremely useful and powerful supplement to such activities, and will help students to see some important connections. Indeed, it is too useful and too powerful to be ignored by continuing to regard the device as mainly of value for dealing with functions and elementary algebra.

References

Australian Education Council (1990). *A National Statement on Mathematics for Australian Schools.* Carlton: Curriculum Corporation.

Kissane, B. (1994). Programmable calculators and mathematics. In Graf, K-D, Malara, N., Zehavi, N. & Ziegenbalg, J. (Eds.) *Proceedings of ICME-7 Working Group 17: Technology in the service of the mathematics curriculum*: *Proceedings of Working Group 17 at ICME-7, Québec 1992* (pp 129-138). Berlin: Freie Universität Berlin.

Kissane, B. (1995). The importance of being accessible: The graphics calculator in mathematics education. In *First Asian Conference on Technology in Mathematics* (pp 161-170). Singapore: The Association of Mathematics Educators. (a) [http://www.edfac.unimelb.edu.au/DSME/TAME/TAME_RESEARCH.html#RES_5]

Kissane, B. (1995) Technology in secondary school mathematics — the graphics calculator as personal mathematical assistant. In Hunting, R., Fitzsimmons, G., Clarkson, P. & Bishop, A. *International Commission on Mathematics Instruction Conference on Regional Collaboration* (pp 383-392). Melbourne: Monash University. (b) [http://www.edfac.unimelb.edu.au/DSME/TAME/TAME_RESEARCH.html#RES_4]

Kissane, B. (1997). Exploring the myths. *The Electronic Classroom, 1*(1), 1. (a)

Kissane, B. (1997). The graphics calculator and the curriculum: The case of probability. In Scott, N. & Hollingsworth, H. (Eds.), *Mathematics: Creating the future* (pp. 397-404). Melbourne: Australian Association of Mathematics Teachers. (b)

Kissane, B. (1997). *Mathematics with a Graphics Calculator*: *Casio fx-7400G.* Perth: Mathematical Association of Western Australia. (c)

Kissane, B. (1997). *More Mathematics with a Graphics Calculator*: *Casio cfx-9850G.* Perth: Mathematical Association of Western Australia. (d)

* This paper was published in the MAV Conference Proceedings:

Kissane, B. 1997, The graphics calculator: Imagine the probabilities, In D. Clarke, P. Clarkson, D. Gronn, M. Horne, L. Lowe, M. Mackinlay & A. McDonough (Eds.) *Mathematics: Imagine the possibilities,* Melbourne, Mathematical Association of Victoria, 487-497.