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Personal technology and the calculus

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The calculus has long been seen as the pinnacle of mathematical endeavour in secondary schools, and rightly so. For as long as most of can remember, algebra and calculus have formed the trunk of mathematics, without which students can progress no further. For the great majority of undergraduates who study mathematics, here and abroad, the first course involves the calculus. In most school systems around the world, the study of the calculus is the jewel in the crown of school mathematics, frequently accessible only to a minority of students in an age group. For many, in fact, the calculus itself is the spectacular jewel in the crown of the mathematics of the past few hundred years

But all is not well in the calculus camp. For some time now, serious misgivings have been voiced about the state of teaching and learning calculus. For example, I recall vividly John Mack, addressing the AAMT Conference in Hobart in 1990, observing that although most of his first year students at his highly selective university were very good at (symbolic) integration, disturbingly few had much idea of what integration actually was or of where, when or why one might want to make use of it. Indeed, many teachers have observed that calculus is regarded by too many students as an extension of algebraic symbolic manipulation; lots of us, including myself, were taught many manipulative techniques and learned them well. But is that really what the calculus is about?

Concern about calculus is not restricted to Australia, of course. In the United States, the Calculus Reform movement is now about ten years old. A recent report (Roberts 1996) was published partly to review the many aspects of calculus reform over a decade. For too many of the undergraduates studying calculus, their present course is also their last mathematics course, in fact. The subject seems not to have much drawing power. Indeed, as Dan Kennedy describes in a delightful article (1995), the trunk that comprises algebra and calculus is like that of a large tree, too difficult to get around and too hard to climb.

Amidst this unease, indeed part of the unease, have been the extraordinary developments in technology over the last two decades. In a mere twenty years, we have gone from rapid dismissal of the idea that computers could deal with symbolic systems such as algebra and calculus, or even that secondary schools might have a computer at all, to the availability of hand-held devices with powerful mathematical capabilities, the most spectacular recent example of which is Texas Instruments' TI-92 (Kissane, 1996). If we are prepared to allow the technology to be desk-bound, computer algebra software such as *Mathematica*, *Maple* and *Derive*, almost undreamt of in the mid 70's, is now commonplace in universities around the world, permitting the symbolic manipulation associated with almost any elementary calculus questions to be efficiently dealt with. To teach the same mathematics in the same way as we have for years in such an environment would require a very powerful argument to be persuasive. As John Kenelly, one of the early Calculus Reformers put it:

> Technology is changing the way we teach. Not because *it's here*, but because *it's everywhere*. Life itself revolves around electronics and we are in the information age. Today's automobiles have more computing devices than the Apollo capsule. Business is a vast network of word processors and spreadsheets. Engineering and Industry are a maze of workstations and automated controls. Our students will have vastly different careers and we, the earlier generation, must radically change the way that education prepares a significantly larger part of the population for information intensive professional lives. (Kenelly, 1996, p24)

But it is not just the development of technology, spectacular and pervasive as it might be, that is important. Rather, it is the development of *personal* technology that cannot be ignored. John Kenelly further notes (emphases in original):

> Only a fool would try to forecast the nature of future technologies. It is frightening to realize

that it was only two decades ago that *Visicalc* spreadsheets on a 64K APPLE II computer were new and exciting things. But if there is one observation to make that might have some hope of surviving, it would be the statement that **powerful hand-held computing units deliver technology to the masses**. Calculators and computers play key and different roles in my classes, and I have always been inclined to try all sorts of new things. But when I look back over my personal experiences, it was the personal/portable nature of the graphics calculator that changed the world for *all* of my students. (Kenelly, 1996, pp26-27)

A technology that is accessible to only some students can be dismissed as a frill, or an optional extra. But a form of technology to which all students have access must be taken seriously. This is the most significant aspect of graphics calculators, as I have argued elsewhere (Kissane 1995a).

In Western Australia, as in many other places, we had around fifteen years experience of the availability of computers to some extent in the teaching and learning of mathematics in schools. In some cases, the computers were used for demonstration, in others, they were accessible in computer laboratories (which were described by Seymour Papert as 'School's defence against the computer'!) But remarkably little changed in the teaching and learning of calculus (or anything else in school mathematics , for that matter) over that time. Now that graphics calculators are used by *all* students studying calculus in WA, and expected to be available and used in examinations as well, the teaching and learning experience has changed markedly, and I expect that so too will some aspects of the official curriculum over the next few years.

One occasionally hears of concerns for 'equity' in discussions about graphics calculators. However, such concerns are frequently misplaced. When technology (such as computers) is 'optional', only more affluent schools and families can afford it. Who else can commit resources to what is optional? However, when *everyone* is expected to make use of technology, the essential inequity of optionality can be removed. (Kissane, Bradley & Kemp 1995). Thus, curiously, a genuine concern for equity would suggest that graphics calculators be made a part of the official curriculum for all, rather than left as a discretionary extra item for the more affluent or the more enthusiastic innovators in schools.

Calculus features of graphics calculators

In considering the relationship between graphics calculators and the calculus, a useful starting point is to inspect the capabilities built in to calculators used by secondary school students. The Casio cfx-9850G is a good example, popular with many students and teachers. The calculator screens in Figures 1 and 2 show some of the numerical calculus capabilities available to a user of this calculator:

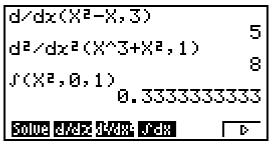

Figure 1: Numerical integration and differentiation

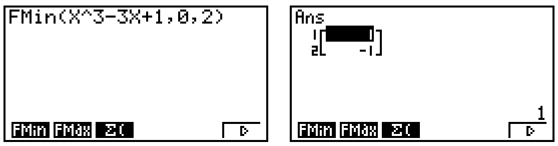

Figure 2: Relative minimum of *f(x) = x^3 - 3x + 1* on (0,2).

As its name suggests, a graphics calculator also has graphic capabilities, and the Casio cfx-9850G has no difficulty drawing graphs of functions, and allowing the graphs to be explored. Figure 3 shows that part of the exploration can involve a derivative trace, displaying both the coordinates of each point as well as the derivative of the function at each point. Clearly, such a facility gives students an opportunity to understand well what information a derivative provides about a curve.

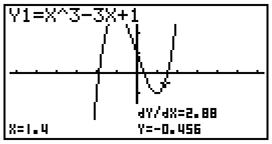

Figure 3: Exploring a drawing of *f(x) = x^3 - 3x + 1*

Graphical explorations are not limited to curve tracing, however. The Casio cfx-9850G includes a number of automatic procedures to locate automatically key aspects of a function, such as relative extrema, intercepts, points of intersection with other graphs and definite integrals. One example, locating the relative minimum of *f(x) = x^3 - 3x + 1* at (1,-1) is shown in Figure 4.

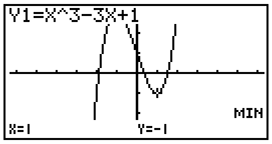

Figure 4: Finding a relative minimum point automatically

As well as graphical information, the cfx-9850G can provide numerical information about a function, including information about its derivatives at tabulated points, as shown in Figure 5. The table of values can be adjusted by the user to start and end at desired points, and the table increment can also be adjusted. Tables can be scrolled with a cursor, to overcome the fact that the screen can show only a few rows and columns at once.

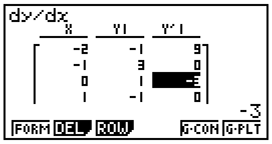

Figure 5: Tabulation of *f(x) = x^3 - 3x + 1* and its derivative

These kinds of capabilities might at first glance give cause for concern among calculus teachers, since they highlight the uncomfortable fact that the calculator can readily provide excellent numerical approximations to many of the numerical answers we are trying to help our students to work out for themselves. It is even feasible that some would regard the use of a this sort of device as a kind of 'cheating', in much the same way that some primary teachers may respond to a four-function calculator's command of arithmetic or some secondary teachers may respond to a scientific calculator with inbuilt commands for solving systems of linear equations.

However, such understandable first reactions are a little short-sighted. In an earlier paper (Kissane, 1995b), I have suggested that a number of metaphors help us to understand the relationships between technology and mathematics, and one of them is certainly the metaphor of 'cheating'. But others include the metaphors of 'laboratory' and of 'tool', each of which applies in the present discussion. Sensible use of graphics calculators with calculus capabilities provides opportunities for students to personally explore and experience the big ideas of calculus in new ways, with considerable promise for educational gain. This perspective is echoed in the opening pages of the MAA publication referred to above:

The case for the use of technology can be summarized this way. In a course where the goal is to teach the calculus, computers or calculators should only be used when there is a sound pedagogical reason for doing so, in which case they should certainly be used. But when used, thought should be given to whether or not they are being used most wisely for the problem at hand. Their proper role is as a tool for experimenting, for discovering, for illustrating, or for substantiating. That is, they are to be used for developing intuition and insight; they should not be used just so the user can crank out answers to ever larger and more complicated exercises. (Roberts, 1996, pp2-3)

A second reaction to these kinds of capabilities is the realisation that the study of calculus can provide much more than numerical answers to questions of practical interest. For example, it potentially provides exact results rather than approximations and deals with general cases rather than specific ones. The calculus allows us to *prove* that there is a single relative minimum values for *f(x) = x^3 - 3x + 1* and that it occurs at the *exact* point for which $x = 1$. It also allows us to examine the general case of *f(x) = x^3 - kx + c.* In the past, students have had no alternative to the calculus to finding answers to questions such as these, and even to curve sketching itself. Indeed, dealing with problems such as finding the minimum value of a function on an interval and sketching a graph of a function have for long been a significant part of the (intrinsic) motivation for studying calculus. When a relatively inexpensive hand-held device can readily answer such practical questions some years before students first enrol in a calculus course, it seems important to re-evaluate the real contribution of the calculus.

Exploring calculus

Space precludes more than a brief sampling of the ways in which a graphics calculator like the Casio cfx-9850G can be used to help students explore the calculus and develop insight into its key ideas. These key ideas involve derivatives, limits, integrals, continuity, asymptotic behaviour and the infinite. More detailed and more extensive suggestions are given elsewhere, such as in Kissane (1997).

Local linearity

A powerful and readily accessible idea with a graphics calculator is that of 'local linearity', the notion that most functions are approximately linear over a sufficiently small interval. On a graphics calculator, students can 'zoom in' repeatedly on a graph of a function to see that the graph becomes approximately linear when the interval becomes small enough. For example, Figure 6 shows the cursor at the same point as in Figure 3, but with a much reduced interval showing on the screen. The 'curve' now appears to be a line - or at least a calculator approximation to one.

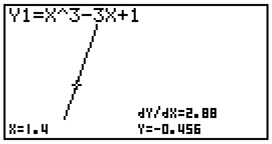

Figure 6: Zooming in on *f(x) = x^3 - 3x + 1* to explore local linearity

Rather than the (difficult) concept of a derivative involving limits of secants and tangents to curves, the notion of local linearity allows for an informal concept of the derivative as the gradient of the *curve itself* to be introduced to and explored by students. This idea, first popularised by David Tall and well used by Mary Barnes in the superb series, *Investigating change* (1992), offers much more potential for insight into the idea of a derivative, than does the formal study of limits, especially for students' first introductions to the calculus.

Derivative functions

While a calculator can evaluate the derivative of a function at a point, a major consequence of this for developing insight is its capacity to do this for many points at once, leading to the idea of a derivative *function*. On the 9850G, either a table or a graph of these is readily produced. Figure 7 shows how to do so graphically, with intuitive commands.

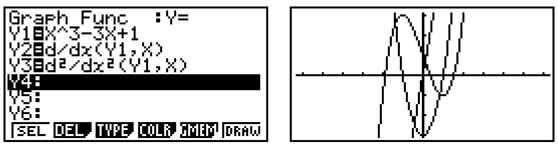

Figure 7: Graphs of *f(x) = x^3 - 3x + 1* and its first two derivative functions

Much can be learned about the idea of a derivative function by exploring such graphs carefully. For example, the *x*-intercepts of the derivative function graph are quite informative about the function itself; similarly, the *x*-intercept of the second derivative function graph is informative about the first derivative function. At a larger level, the apparent quadratic graph for the first derivative function and the apparent linear graph for the second derivative function are insightful, given that the original function is cubic. While a graphics calculator is a lot slower and the screen a lot chunkier than a large computer, and (in the case of the cfx-9850) is restricted to only three colours for graphs, its portability and accessibility to students make explorations of these kinds possible.

Another view of derivative functions

The limiting secant definition of the derivative of a function allows another kind of graphical exploration of derivatives. To illustrate, Figure 8 shows the graphs of $f(x)$ = sin x and the approximation to its derivative function, $g(x)$ = (sin (x + 0.1) - sin x) ÷ 0.1.

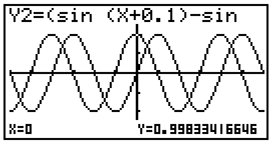

Figure 8: Approximating a derivative function

Even though 0.1 is clearly not 'infinitesimally small', the approximation is remarkably close to the derivative function $f(x)$ = cos x. Of course, students can explore the limiting process readily by selecting smaller and smaller values to replace 0.1, all the while developing a sense of the limiting process as well as the result itself, which is surely counter-intuitive and unexpected to a beginner. Even most mathematics teachers are surprised at how quickly the limiting process converges.

Limits

The zooming capabilities of graphics calculators, both graphically and numerically, allow a good intuitive sense of the difficult concepts of limits to be explored by students. One example is shown above in Figure 8. As a second example, consider the limit of sin x/x as x tends to zero. Whether using a graph or a table, Figure 9 suggests mechanisms for students to explore this limit by studying the relevant function near x = 0.

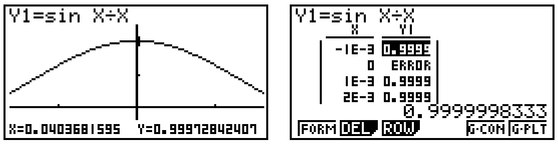

Figure 9: Exploring a limit, graphically and numerically

Although the value is not defined at x = 0 (as shown in the table), the calculator allows students to experience the limiting process in ways not normally available to them.

Conclusion

This paper has provided only a few examples from a potentially much larger set. But anyway a paper is a poor substitute for the experience of exploring the ideas on a calculator for yourself, an observation no less true for teachers than it is for their students. The examples support the proposition that teaching and learning calculus can be influenced positively by taking advantage of the capabilities built into graphics calculators such as the Casio cfx-9850G. Personal technology of this kind has very much more potential to significantly influence both teaching and curriculum in the calculus than did the less personal technologies of computers. It is already exceedingly difficult to argue, as the millenium draws to a close, that calculus ought to be taught and learned as if the technology had not been invented, or was not affordable to schools and their students.

References

Barnes, M. 1992, *Investigating change*, Melbourne, Curriculum Corporation.

Kenelly, J. 1996, Technology in mathematics instruction, in Roberts, A. W. (ed) 1996, *Calculus: The dynamics of change*, Washington, DC, Mathematical Association of America, 24-27.

Kennedy, D. 1995, Climbing around on the tree of mathematics, *Mathematics Teacher*, 88, 6, 460-465.

Kissane, B. 1997, *More mathematics with a graphics calculator: Casio cfx-9850G*, Perth, Mathematical Association of Western Australia.

Kissane, B. 1995, The importance of being accessible: The graphics calculator in mathematics education, in *First Asian Conference on Technology in Mathematics* (pp 161-170), Singapore: The Association of Mathematics Educators. (a) [http://www.edfac.unimelb.edu.au/DSME/TAME/TAME_RESEARCH.html#RES_5]

Kissane, B. 1995, Technology in secondary school mathematics - the graphics calculator as personal mathematical assistant. In Hunting, R., Fitzsimmons, G., Clarkson, P. & Bishop, A. *International Commission on Mathematics Instruction Conference on Regional Collaboration* (pp 383-392), Melbourne: Monash University. (b) [http://www.edfac.unimelb.edu.au/DSME/TAME/TAME_RESEARCH.html#RES_4]

Kissane, B., Bradley, J. & Kemp, M. 1994, Graphics calculators, equity and assessment, *Australian Senior Mathematics Journal*, 8(2), 31-43.

Kissane, B. 1996, The Texas Instruments TI-92 ... A handful of mathematics, *Australian Senior Mathematics Journal*, 10, 2, 51-61.

Roberts, A. W. (ed) 1996, *Calculus: The dynamics of change*, Washington, DC, Mathematical Association of America.