Algorithms to Compute the Lyndon Array*

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Abstract. In the Lyndon array \( \lambda = \lambda_x[1..n] \) of a string \( x = x[1..n] \), \( \lambda[i] \) is the length of the longest Lyndon word starting at position \( i \) of \( x \). The computation of \( \lambda \) has recently become of great interest, since it was shown (Bannai et al., The “Runs” Theorem [2]) that the runs in \( x \) are computable in linear time from \( \lambda_x \). Here we first describe three algorithms for computing \( \lambda_x \) that have been suggested in the literature, but for which no structured exposition has been given. Two of these algorithms execute in \( O(n^2) \) time in the worst case; the third achieves \( \Theta(n) \) time, but at the expense of prior computation of both the suffix array and the inverse suffix array of \( x \). We then go on to describe two variants of a new algorithm that avoids prior computation of global data structures and executes in worst-case \( O(n \log n) \) time. Experimental evidence suggests that all but one of these five algorithms require only linear execution time in practice, with the two new algorithms faster by a small factor. We conjecture that there exists a fast and worst-case linear-time algorithm to compute the Lyndon array that is also “elementary” (making no use of global data structures such as the suffix array).

1 Introduction

If \( x = uv \) for some \( u \) and nonempty \( v \), then \( vu \) is said to be the \( |u|^{\text{th}} \) rotation of \( x \), written \( vu = R_{|u|}(x) \). If there exists a string \( u \) and an integer \( e > 1 \)

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such that \( x = u^e \), then \( x \) is said to be a repetition; otherwise \( x \) is primitive. A primitive string \( x \) that is lexicographically least among all its rotations \( R_k(x), k = 0, 1, \ldots, |x| - 1 \), is said to be a Lyndon word.

The Lyndon array \( \lambda = \lambda_x[1..n] \) (equivalently, \( L = L_x[1..n] \)) of a given nonempty string \( x = x[1..n] \) gives at each position \( i \) the length (equivalently, the end position) of the longest Lyndon word starting at \( i \):

\[
\begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
x = a & b & a & a & b & a & a & b \\
\lambda = 2 & 1 & 5 & 2 & 1 & 2 & 1 & 3 & 2 & 1 \\
L = 2 & 2 & 7 & 5 & 5 & 7 & 7 & 10 & 10 & 10
\end{array}
\]

The Lyndon array has recently become of interest since Bannai et al. [2] showed that it could be used to efficiently compute all the maximal periodicities (“runs”) in a string. In this paper we describe four algorithms to compute \( \lambda_x \), three of them shown experimentally to be running in \( \Theta(n) \) time in practice. Section 2 makes various observations that apply generally to the Lyndon array and its computation. In Section 3 we describe three algorithms, two that require \( O(n^2) \) time in the worst case, of which one is very fast and apparently linear in practice, the other supralinear in practice and \( O(n \log n) \) in the average case on binary strings. The third algorithm is simple and worst-case linear-time, but requires suffix array construction and so is a little slower. Section 4 describes two variants of a new algorithm that uses only elementary data structures (no suffix arrays). One variant is \( O(n^2) \) in the worst case, the other guarantees \( O(n \log n) \) time, but with no clear advantage in processing time. Section 5 describes the results of preliminary experiments on the algorithms; Section 6 outlines future work.

2 Preliminaries

Here we make various observations that apply to the algorithms described below.

**Observation 1** Let \( x = w_1 w_2 \cdots w_k \) be the Lyndon decomposition \([5, 9]\) of \( x \), with Lyndon words \( w_1 \geq w_2 \geq \cdots \geq w_k \). Then every Lyndon word \( x[i..L[i]] \) of length \( \lambda[i] \) is a substring of some \( w_h, h \in 1..k \).

**Proof.** For some \( h \in 1..k-1 \), consider \( w_h \) with nonempty proper suffix \( v_h \), and for some \( t \in 1..k-h \), consider \( w_{h+t} \) with nonempty prefix \( u_{h+t} \). Since \( w_h \) is a Lyndon word, \( w_h < v_h \), and by lexorder, \( u_{h+t} \leq w_{h+t} \). Thus \( v_h > w_h \geq w_{h+t} \geq u_{h+t} \), and so \( v_h w_{h+1} \cdots w_{h+t-1} u_{h+t} \) cannot be a Lyndon word for any choice of \( h \) or \( t \).

Therefore to compute \( L_x \) it suffices to consider separately each distinct element \( w_h \) in the Lyndon decomposition of \( x \). Hence, without loss of generality suppose \( x \) is a Lyndon word and write it in the form \( x_1 x_2 \cdots x_m \), where for each \( r \in 1..m \), \( |x_r| = \ell_r \) and

\[
x_r[1] \leq x_r[2] \leq \cdots \leq x_r[\ell_r],
\]

(2)
while for \( 1 \leq r < m \),
\[
x_r[\ell_r] > x_{r+1}[1].
\]
(3)

We call \( x_r \) a **range** in \( x \) and the boundary between \( x_r \) and \( x_{r+1} \) a **drop**. We identify a position \( j \) in range \( x_r \), \( 1 \leq j \leq \ell_r \), with its equivalent position \( i \) in \( x \) by writing \( i = S_{r,j} = \sum_{r'=1}^{r-1} \ell_{r'} + j \).

**Observation 2** Let \( i = S_{r,j} \) be a position in \( x \) that corresponds to position \( j \) in range \( x_r \).

(a) If \( x_r[j] = x_r[\ell_r] \), then \( L[i] = i \).

(b) Otherwise, \( L[i] = i' \), where \( i' \) is the final position in some range \( x_{r'} \), \( r' \geq r \); that is, \( i' = \sum_{s=1}^{r'} \ell_s \).

Proof. (a) is an immediate consequence of (2) and (3). To prove (b), suppose that \( x[i..L[i]] \) is a maximum-length Lyndon word, where \( L[i] \) falls within range \( r' \) but \( L[i] < i' \). Since by (2) \( x[L(i)] \leq x[L[i]+1] \), there are two consecutive Lyndon words \( x[i..L[i]], x[L[i]+1] \) that by the Lyndon decomposition theorem [5] can be merged into a single Lyndon word \( x[i..L[i]+1] \). Thus \( x[i..L[i]] \) is not maximum-length, a contradiction.

We see then that if \( x_r[j] < x_r[\ell_r] \), then \( x_r[j..\ell_r] \) is a (not necessarily maximum-length) Lyndon word, and for \( i = S_{r,j} \), \( L[i] \geq S_{r,\ell_r} \):

\[
\begin{align*}
1 & 2 3 4 5 6 7 8 9 10 11 12 13 \\
x = & a a b | a a b | a b | a b | a b | b \\
L = & 13 13 4 4 9 7 7 9 9 13 13 12 13
\end{align*}
\]

(4)

More generally, the vectors \((i, L[i])\) satisfy a “Monge” property that is exploited by Algorithm NSV (Section 4):

**Observation 3** Suppose positions \( i, j \) in \( x[1..n] \) satisfy \( 1 \leq i < j \leq n \). Then either \( L[i] \leq j \) or \( L[i] \geq L[j] \); the vectors \((i, L[i])\) and \((j, L[j])\) are nonintersecting.

Proof. Suppose two such vectors do intersect. Then the maximum-length Lyndon words \( w_1 = x[i..L[i]] \) and \( w_2 = x[j..L[j]] \) have a nonempty overlap, so that we can write \( w_1 = uv \), \( w_2 = vv' \) for some nonempty \( v \). But then, by well-known properties of Lyndon words, \( w_1 < v < w_2 < v' \), implying that \( w_1v' \) is a Lyndon word, contradicting the assumption that \( w_1 \) is maximum-length.

Expressing a string in terms of its ranges has the same useful lexicographic property that writing it in terms of its letters does:

**Observation 4** Suppose strings \( x \) and \( y \) are expressed in terms of their ranges: \( x = x_1x_2 \cdots x_m \), \( y = y_1y_2 \cdots y_n \). Suppose further that for some least integer \( r \in \{1, \min(m,n)\} \), \( x_r \neq y_r \). Then \( x < y \) (respectively, \( x > y \)) according as \( x_r < y_r \) (respectively, \( x_r > y_r \)).
Proof. If \( x_r < y_r \), then either

(a) \( x_r \) is a nonempty proper prefix of \( y_r \); or
(b) there is some least position \( j \) such that \( x_r[j] < y_r[j] \).

In case (a), if \( r = m \), then \( x \) is actually a prefix of \( y \), so that \( x < y \), while if \( r < m \), then by (3), \( x_{r+1}[1] < y_r[x_r]+1 \), and again \( x < y \). In case (b) the result is immediate. The proof for \( x_r > y_r \) is similar.

3 Basic Algorithms

Here we outline three algorithms for which no clear exposition is available in the literature. We remark that the Lyndon array computation is equivalent to “Lyndon bracketing”, for which an \( O(n^2) \) algorithm has been described [19].

3.1 Folklore — Iterated MaxLyn

For a string \( x \) of length \( n \), recall that the prefix table \( \pi[1..n] \) is an integer array in which for every \( i \in 1..n \), \( \pi[i] \) is the length of the longest substring beginning at position \( i \) of \( x \) that matches a prefix of \( x \). Given a nonempty string \( x \) on alphabet \( \Sigma \), let us define \( x' = x\$ \), where the sentinel \( \$ < \mu \) for every letter \( \mu \in \Sigma \).

**Observation 5** \( x \) is a Lyndon word if and only if for every \( i \in 2..n \), \( x'[1+k] < x'[i+k] \), where \( k = \pi[i] \).

This result forms the basis of the algorithm given in Figure 1 that computes the length \( \max \in 1..n-j+1 \) of the longest Lyndon factor at a given position \( j \) in \( x[1..n] \). Its efficiency is a consequence of the instruction \( i \leftarrow i + k + 1 \) that skips over positions in the range \( i + 1..i + k - 1 \), effectively assuming that for every position \( i^* \) in that range, \( i^* + \pi[i^*] \leq i + k \). Lemma 11, given in Appendix 1, justifies this assumption. Simply repeating MaxLyn at every position \( j \) of \( x \) gives a simple, fast \( O(n^2) \) time and \( O(1) \) additional space algorithm to compute \( \lambda x \).

Recent work on the prefix table [4, 6] has confirmed its importance as a data structure for string algorithms. In this context it is interesting to find that Lyndon words \( x \) can be characterized in terms of \( \pi x \):

**Observation 6** Suppose \( x = x[1..n] \) is a string on alphabet \( \Sigma \) such that \( x[1] \) is the least letter in \( x \). Then \( x \) is a Lyndon word over \( \Sigma \) if and only if for every \( i \in 2..n \),

(a) \( i + \pi x[i] < n + 1 \); and
(b) for every \( j \in i + 1..i + \pi x[i] - 1, j + \pi x[j] \leq i + \pi x[i] \).
procedure MaxLyn(x[1..n], j, Σ, <) : integer
i ← j + 1; max ← 1
while i ≤ n do
    k ← 0
    while x′[j + k] = x′[i + k] do
        k ← k + 1
    if x′[j + k] < x′[i + k] then
        i ← i + k + 1; max ← i - 1
    else
        return max
Fig. 1. Algorithm MaxLyn

3.2 Recursive Duval Factorization: Algorithm RDuval

Rather than independently computing the maximum-length Lyndon factor at each position $i$, as MaxLyn does, Algorithm RDuval recursively computes the Lyndon decomposition into maximum factors, at each step taking advantage of the fact that $L[i]$ is known for the first position $i$ in each factor, then recomputing with the first letters removed. By Observation 1, whenever $x = x[1..n]$ is a Lyndon word, we know that $L[1] = n$. Thus computing the Lyndon decomposition $x = w_1 w_2 \cdots w_k$, $w_1 \geq w_2 \geq \cdots \geq w_k$, allows us to assign $\lambda[i_j] = |w_j|$, where $i_j$ is the first position of $w_j$, $j = 1, 2, \ldots, k$.

Algorithm RDuval applies this strategy recursively, by assigning $\lambda[i_j] ← |w_j|$, then removing the first letter $i_j$ from each $w_j$ to form $w'_j$, to which the Lyndon decomposition is applied in the next recursive step. This process continues until each Lyndon word is reduced to a single letter.

The asymptotic time required for RDuval is bounded above by $n$ times the maximum depth of the recursion, thus $O(n^2)$ in the worst case — consider, for example, the string $x = a^{n-1}b$. However, to estimate expected behaviour, we can make use of a result of Bassino et al. [3]. Given a Lyndon word $w$, they call $w = uv$ the standard factorization of $w$ if $u$ and $v$ are both Lyndon words and $v$ is of maximum size. They then show that if $w$ is a binary string ($\Sigma = \{a, b\}$), the average length of $v$ is asymptotically $3|w|/4$. Thus each recursive application of RDuval yields a left Lyndon factor of expected length $|w|/4$ and a remainder of length $3|w|/4$ to be factored further. It follows that the expected number of recursive calls of RDuval is $O(\log_{4/3} n)$. Hence

Lemma 7 On binary strings RDuval executes in $O(n \log_{4/3} n)$ time on average.

Example 8 For

\[
x = \begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
\end{array}
\]
\[
\lambda = \begin{array}{cccccccccccc}
12 & 2 & 1 & 9 & 3 & 1 & 1 & 3 & 1 & 1 & 2 & 1 \\
\end{array}
\]

the factors considered are first 1–12, then
3.3 NSV Applied to the Inverse Suffix Array

The idea of the "next smaller value" (NSV) array for a given array (string) \( x \) has been proposed in various forms and under various names \([1, 10, 17, 12]\).

**Definition 9 (Next Smaller Value)** Given an array \( x[1..n] \) of ordered values, \( NSV = NSV_x[1..n] \) is the next smaller value array of \( x \) if and only if for every \( i \in 1..n \), \( NSV[i] = j \), where

\[(a) \text{ for every } h \in 1..j-1, x[i] \leq x[i+h]; \text{ and}
(b) \text{ either } i+j = n+1 \text{ or } x[i] > x[i+j].\]

**Example 10**

\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\end{array}
\]

\[
x = 3 8 7 10 2 1 4 9 6 5
\]

\[
NSV_x = 4 1 2 1 1 5 4 1 1 1
\]

As shown in various contexts in \([12]\), \( NSV_x \) can be computed in \( \Theta(n) \) time using a stack. Our main observation here, touched upon in \([13]\), is that \( \lambda_x \) can be computed merely by applying NSV to the inverse suffix array \( ISA_x \). Proof of this claim can be found in Appendix 2; here we present the very simple \( \Theta(n) \)-time, \( \Theta(n) \)-space algorithm for this calculation:

```plaintext
procedure NSVISA(x[1..n]) : \( \lambda_x[1..n] \\
Compute \( SA_x \) \hspace{1em} \textbf{(see }[16, 18] )
Compute \( ISA_x \) from \( SA_x \) in place \hspace{1em} \textbf{(see }[18] )
\( \lambda_x \leftarrow \text{NSV}(ISA_x) \) \hspace{1em} \text{(in place)}
```

**Fig. 2.** Apply NSV to \( ISA_x \)

4 Elementary Computation of \( \lambda_x \) Using Ranges

In this section we describe an approach to the computation of \( \lambda_x \) that applies a variant of the NSV idea to the ranges of \( x \). Figure 3 gives pseudocode for Algorithm NSV* that uses the NSV stack \( ACTIVE \) to compute \( \lambda \). The processing identifies ranges in a single left-to-right scan of \( x \), making use of two range comparison routines, COMP and MATCH. COMP compares adjacent individual
ranges \( x_r \) and \( x_{r+1} \), returning \( \delta_1 = -1, 0, +1 \) according as \( x_r < x_{r+1}, x_r = x_{r+1}, x_r > x_{r+1} \). MATCH similarly returns \( \delta_2 \) for adjacent sequences of ranges; that is,

\[
X_r = x_rx_{r+1} \cdots x_{r+s}, \text{ for some } s \geq 1;
X_{r+s+1} = x_{r+s+1}x_{r+s+2} \cdots x_{r+t}, \text{ for some } t \geq 1.
\]

Algorithm NSV* is based on the idea encapsulated in Lemma 15 of Appendix 2, the main basis of the correctness of Algorithm NSVISA. We process \( x \) from left to right, using a stack ACTIVE initialized with index 1. At each iteration, the top of the stack (say, \( j \)) is compared with the current index (say, \( i \)). In particular, we need to compare \( s_\lambda_x(i) \) with \( s_\lambda_x(j) \), where \( s_\lambda_x(i) = x[i..n] \). As long as \( s_\lambda_x(i) \geq s_\lambda_x(j) \), NSV* pushes the current index and continues to the next. When \( s_\lambda_x(i) < s_\lambda_x(j) \), it pops the stack and puts appropriate values in the corresponding indices of \( \lambda_x \). As noted above, especially Observations 1–3, ranges are employed to expedite these suffix comparisons.

Two auxiliary arrays, nextequal and period, are required to handle situations in which MATCH finds that a suffix of a previous range at position \( j \) equals the current range at position \( i \). Thus, when \( \delta_2 = 0 \), the algorithm assigns nextequal[\( i \)] \( \leftarrow \) \( j \) before \( i \) is pushed onto ACTIVE. Then when a later MATCH yields \( \delta_2 = 0 \), the value of period — that is, the extent of the following periodicity — may need to be set or adjusted, as shown in the following example:

\[
\begin{align*}
x & = a \ a \ a \ b \ a \ a \ b \ a \ a \ b \ a \ b \\
\text{nextequal} & = 0 \ 5 \ 0 \ 0 \ 8 \ 0 \ 0 \ 11 \ 0 \ 0 \ 14 \ 0 \ 0 \ 0 \\
\text{period} & = 0 \ 12 \ 0 \ 0 \ 9 \ 0 \ 0 \ 6 \ 0 \ 0 \ 4 \ 0 \ 0 \ 0
\end{align*}
\]

A straightforward implementation of COMP and MATCH could require a number of letter comparisons equal to the length of the shorter of the two sequences of ranges being matched. However, by performing \( \Theta(n) \)-time preprocessing, we can compare two ranges in \( O(\sigma) \) time, where \( \sigma = |\Sigma| \) is the alphabet size. Given \( \Sigma = \{\mu_1, \mu_2, \ldots, \mu_\sigma\} \), we define Parikh vectors \( P_r[1..\sigma] \), where \( P_r[j] \) is the number of occurrences of \( \mu_j \) in range \( x_r \). Since ranges are monotone nondecreasing in the letters of the alphabet, it is easy to compute all the \( P_r, r = 1, 2, \ldots, m, \) in linear time in a single scan of \( x \). Similarly, during the processing of each range \( x_r \), any value \( P_{r,j} \), the Parikh vector of the suffix \( x_r[j..r] \), can be computed in constant time for each position considered. Thus we can determine the lexicographical order of any two ranges (or part ranges) \( x_r \) and \( x_r \) in \( O(\sigma) \) time rather than time \( O(\max(\ell_r, \ell_{r'}) ) \). The variant of NSV* that uses Parikh vectors is called PNSV*; otherwise NPNSV* for Not Parikh.

In Appendix 3 we describe briefly another approach to this suffix comparison problem, which also achieves run time \( O(n \log n) \) by maintaining a simple data structure requiring \( O(n \log n) \) space.

Now consider the worst case behaviour of Algorithm NSV*. Given the initial string \( x_0 = a^hba^hc_0, h \geq 1, c_0 > b > a \), let \( x_k^{(h)} = x_k = x_{k-1}x_{k-1}^*, k = 1, 2, \ldots, \) with \( x_{k-1}^* \) identical to \( x_{k-1} \) except in the last position, where the letter
procedure NSV* (x, λ)
nextequal ← 0n; period ← 0n
push(ACTIVE) ← 1
▷ x[n+1] = $, a letter smaller than any in Σ.
for i ← 2 to n+1 do
  prev ← 0; j ← peek(ACTIVE)
  ▷ COMP compares suffixes specified by i, j of two ranges.
  δ1 ← COMP(x[j], x[i]); δ2 ← 1
  while (δ1 ≥ 0 and δ2 > 0) do
    if δ1 = 0 then δ2 ← MATCH(x[j], x[i])
    if δ2 > 0 then
      if prev = 0 or nextequal[j] ≠ prev then λ[j] ← i − j
      else
        λ[j] ← offset ← prev − j
        if period[prev] = 0 then
          if λ[prev] > offset then
            λ[j] ← λ[j] + λ[prev]
        else
          if nextequal[j] = prev and offset ≠ λ[prev] then
            λ[j] ← λ[j] + period[prev]
          if λ[prev] = offset then
            ▷ Current position is a part of periodic substring
            if period[prev] = 0 then
              period[j] ← period[prev] + 2 × offset
            else
              period[j] ← period[prev] + offset
        pop(ACTIVE)
        prev ← j; j ← peek(ACTIVE)
      ▷ Empty stack implies termination.
      if j = 0 then EXIT
      δ1 ← COMP(x[j], x[i])
    ▷ Finished processing i — it goes to stack.
    if δ2 = 0 then nextequal[j] ← i
  push(ACTIVE) ← i

Fig. 3. Computing λx using modified NSV

c_k > c_{k-1} replaces c_{k-1}. Then x_k has length n = (h+1)m, where m = 2^{k+1} is the number of ranges in x_k. In Appendix 4 it is shown in Lemma 16 that x_k is a worst-case input for Algorithm NSV*, which requires O(n log n) range matches in such cases. Since PNSV* compares two ranges in O(σ) time, it therefore requires O(σn log n) time in the worst case, thus O(n log n) for constant σ. In Appendix 4 we argue that NPNSV* is also O(n log n) in the worst case.
5 Experimental Results

We have done preliminary tests on the algorithms described above, including the two variants of NSV*. The equipment used was an Intel(R) Core i3 at 1.8GHz and 4GB main memory under a 64-bit Windows 7 operating system. Figure 4 shows the results of exhaustive tests of the algorithms on all binary strings of lengths 11–22, with all but RDuval displaying linear-time behaviour. MaxLyn and NPNSV* are roughly equivalent in time requirement, with NSVISA several times slower, PNSV* perhaps 10 times slower.

We have also tested the linear average-case algorithms on much longer binary strings, several megabytes in length, both random and highly periodic [11]. On random strings, PNSV* and NPNSV* are comparable in speed and fastest by a factor of 2 or 3, while on the periodic strings, MaxLyn has an advantage by approximately the same margin. More testing needs to be done, especially on strings defined on larger alphabets, but of the current collection, it appears that the two new $O(n \log n)$-time algorithms are the algorithms of choice.

![Fig. 4. Five algorithms compared on all binary strings of lengths $n \in 11..22$: the average processing time for each $n$ is given in $10^{-4}$ seconds.](image)

6 Future Work

There is reason to believe [15] that the Lyndon array computation is less hard than suffix array construction. Thus the authors conjecture that there is a linear-time elementary algorithm (no suffix arrays) to compute the Lyndon array.
References


Appendix 1

The following result justifies the strategy employed in Algorithm MaxLyn (Figure 1):

**Lemma 11** Suppose that for some position $i$ in a Lyndon word $x[1..n]$, $k = \pi[i] \geq 2$. Then for every $j \in i + 1 \ldots i + k - 1$, $\pi[j] \leq i + k - j$.

**Proof.** The result certainly holds for $i + k = n + 1$, so we consider $i + k \leq n$. Assume that for some $j \in i + 1 \ldots i + k - 1$, $\pi[j] > i + k - j$. It follows that

$$x[1..i + k - j + 1] = x[j..i + k],$$

while $x[j - i + 1..k] = x[j..i + k - 1]$. Since $x$ is Lyndon, therefore $x[1 + k] \prec x[i + k]$, and so we find that

$$x[j - i + 1..1 + k] \prec x[j..i + k].$$

From (5) and (6) we see that $x[1..k + 1]$ has suffix $x[j - i + 1..1 + k]$ satisfying $x[j - i + 1..k + 1] \prec x[1..i + k - j + 1]$, contradicting the assumption that $x$ is Lyndon.

Appendix 2

Here we prove Theorem 12 that justifies Algorithm 2:

**Theorem 12** For a given string $x = x[1..n]$ on alphabet $\Sigma$, totally order by $\prec$, let $ISA = ISA_x^\prec$. Then for every $i \in 1..n$, the substring $x[i..j]$ is a longest Lyndon factor with respect to $\prec$ if and only if

(a) for every $h \in i + 1..j$, $ISA[j] < ISA[h]$; and
(b) either $j = n$ or $ISA[j + 1] < ISA[i]$.

The following well-known result is needed to prove Lemma 14:

**Lemma 13 (Duval, Lemma 1.6, [9])** Suppose $x \in \Sigma^+$, where $\Sigma$ is an alphabet totally ordered by $\prec$. Let $x = u^r u_1 b$, where $u$ is nonempty, $r \geq 1$, $u_1$ a possibly empty proper prefix of $u$, and the letter $b \neq u|[u_1]| + 1$.

(a) If $b \prec u|[u_1]| + 1$, then $u$ is a longest Lyndon prefix of $xy$ for any $y$;
(b) if $b \succ u|[u_1]| + 1$, then $x$ is Lyndon with respect to $\prec$.

For a given string $x[1..n]$, let $s_x(i) = x[i..n]$ denote the suffix of $x$ beginning at position $i$. When clear from context we write just $s(i)$.

**Lemma 14** Consider a string $x = x[1..n]$ over alphabet $\Sigma$ totally ordered by $\prec$. Let $x[i..j]$ be the longest Lyndon factor of $x$ starting at $i$. Then $s_x(i) \prec s_x(k)$ for every $k \in i + 1..j$ and either $j = n$ or $s_x(j + 1) < s_x(i)$.
Thus we want to show that \( s \) and we reformulate Lemma 15 in terms of the inverse suffix array \( ISA \) of \( x \).

**Proof.** Because \( x[i\ldots j] \) is Lyndon, therefore for any \( i < k \leq j \), \( x[i\ldots j] \prec x[k\ldots j] \) and so \( s(i) \prec s(k) \). If \( j = n \), we are done. So we may assume \( j < n \), and we want to show that \( s(j+1) \prec s(i) \). Suppose then that \( s(j+1) \not\prec s(i) \). Since \( s(i) \) and \( s(j+1) \) are distinct, it follows that \( s(i) \prec s(j+1) \). If we let \( d = \text{lcp}(s(i), s(j+1)) + 1 \), two cases arise:

(a) \( 0 \leq d \leq j - i \). 

Here \( i \leq i + d \leq j \). Thus \( x[i\ldots i+d-1] = x[j+1\ldots j+d] \) and \( x[i+d] \prec x[j+1+d] \), and so for \( j < k \leq j+1+d \), \( x[i\ldots j+1+d] \prec x[k\ldots j+1+d] \). Since \( x[i\ldots j] \) is Lyndon, \( x[i\ldots j] \prec x[k\ldots j] \) and so \( x[i\ldots j+1+d] \prec x[k\ldots j+1+d] \) for any \( i < k \leq j \). Thus \( x[i\ldots j+1+d] \) is Lyndon, contradicting the assumption that \( x[i\ldots j] \) is the longest Lyndon factor starting at \( i \).

(b) \( 0 < j - i \leq d \).

Let \( d = r(j-i) + d_1 \), where \( 0 \leq d_1 < j - i \). Then \( r \geq 1 \) and \( x[i\ldots j+1+d] = u^ru_1b \) where \( u = x[i\ldots j] \), 

\[ u_1 = x[j+r(j-i)+1\ldots j+r(j-i)+d_1-1] = x[j+r(j-i)+1\ldots j+d-1] \]

is a prefix of \( x[i\ldots j] \), and \( x[i+d] \prec x[j+1+d] \), so that by Lemma 13 (b), \( x[i\ldots j+1+d] \) is Lyndon, contradicting the assumption that \( x[i\ldots j] \) is the longest Lyndon factor starting at \( i \).

Thus \( s(j+1) \prec s(i) \), as required.

Lemma 15 describes the property of being a longest Lyndon factor of a string \( x \) in terms of relationships between corresponding suffixes.

**Lemma 15** Consider a string \( x = x[1\ldots n] \) over an alphabet \( \Sigma \) with an ordering \( \prec \). A substring \( x[i\ldots j] \) is a longest Lyndon factor of \( x \) with respect to \( \prec \) if and only if \( s_x(i) \prec s_x(k) \) for every \( k \in i+1..j \) and either \( j = n \) or \( s_x(j+1) \prec s_x(i) \).

**Proof.** Let (A) denote \{ \( x[i\ldots j] \) is a longest Lyndon factor of \( x \) \} and let (B) denote \{ \( s(i) \prec s(k) \) for any \( 1 \leq k \leq j \) and \( s(j+1) \prec s(i) \) \}. Then (A) \( \Rightarrow \) (B) follows from Lemma 14, so we need to prove that (B) \( \Rightarrow \) (A).

Suppose then that (B) holds, and let \( x[i\ldots k] \) be a longest Lyndon factor of \( x \) starting at position \( i \). If \( k < j \), then by Lemma 14, \( s(k+1) \prec s(i) \), a contradiction since \( k+1 \leq j \). If \( k > j \), then by Lemma 14, \( s(i) \prec s(j+1) \) because \( j+1 \leq k \), which again gives us a contradiction. Thus \( k = j \) and \( x[i\ldots j] \) is a longest Lyndon factor of \( x \).

Now we reformulate Lemma 15 in terms of the inverse suffix array \( ISA \) of \( x \) using the relationship that \( s(i) \prec s(j) \iff ISA[i] \prec ISA[j] \), thus yielding Theorem 12, as required. Hence the Lyndon array can be computed in a simple three-step algorithm, as shown in Figure 2, that executes in \( \theta(n) \) time and uses only one additional array of integers.
Appendix 3

Here we describe a simple data structure that yields an alternative approach to Algorithm NSV*, based on the comparison of longest Lyndon factors as described in Lemma 15. The dictionary of basic factors of string \( x[1..n] \) consists of a sequence of arrays \( D_t, 0 \leq t \leq \log n \). The array \( D_t \) records information about factors of \( x \) of length \( 2^t \) — that is, the basic factors. In particular, \( D_t[i] \) stores the rank of \( x[i..i + 2^t - 1] \), so that

\[
x[i..i + 2^t - 1] \preceq x[i..i + 2^t - 1] \iff D_t[i] \leq D_t[i].
\]

This dictionary requires \( O(n \log n) \) space and can be constructed in \( O(n \log n) \) time as follows. \( D_0 \) contains information about consecutive symbols of \( x \) and hence can be computed in \( O(n \log n) \) time by sorting all the symbols appearing in \( x \) and mapping them to numbers from 1 and onward. Once \( D_t \) is computed, we can easily compute \( D_{t+1} \) by spending \( O(n) \) time on a radix sort, because \( u[i..i + 2^{t+1} - 1] \) is in fact a concatenation of the factors \( u[i..i + 2^t - 1] \) and \( u[i + 2^t..i + 2^{t+1} - 1] \).

Once this dictionary is computed, we can compare any two factors by comparing two appropriate overlapping basic factors (i.e., factors having length power of two), which is done by checking the corresponding \( D \) array from the dictionary. This will require constant time and hence each suffix-suffix comparison can be done in constant time.

Appendix 4

Lemma 16 Let \( RM^{(h)}(k) \) denote the number of range matches needed by Algorithm NSV* to compute \( \lambda x_k \) of length \( n = (h+1)m \), where \( m = 2^{k+1} \) is the number of ranges in \( x_k \). Then \( RM^{(h)}(k) = m(\log_2 m - 1) + 1 \in \Theta(n \log n) \).

Proof. Consider the rightmost two ranges \( s_0 = a^h b a^h c_k \) of \( x_k \). NSV* requires one range match to discover that \( a^h b < a^h c_k \), which we may denote by the vector \((1, 0)\) that associates the one match with the leftmost of the two ranges being compared. Similarly, with the rightmost four ranges \( s_1 = a^h b a^h c_0 a^h b a^h c_k \) of \( x_k \), we may associate the vector \((2, 2, 1, 0)\), counting a maximum two more range matches performed by NSV* on each of \( a^h b \) and \( a^h c_0 \) with ranges to their right. Observe that as the vector is extended to the left, the existing elements are unchanged. Now consider the four ranges \( a^h b a^h c_0 a^h b a^h c_{k-1} \) that form the prefix of \( s_2 \) on the left of \( s_1 \). It is easy to see that the maximum number of range matches associated with the start positions of these four ranges can be counted \((3, 3, 3, 3)\), thus extending the vector to \((3, 3, 3, 3, 2, 2, 1, 0)\). The next eight positions on the left will yield a maximum \((4, 4, 4, 4, 4, 4, 4, 4)\) range matches, and so on, until the
beginning of $x_k$ is reached. Thus

$$RM^{(h)}(k) = \sum_{j=0}^{k} (j+1)2^j$$

$$= \sum_{j=1}^{k} j2^j + \sum_{j=0}^{k} 2^j$$

$$= (k(2^{k+2} - 2^{k+1}) - 2^{k+1} + 2) + (2^{k+1} - 1) \text{ [14, p. 33]}$$

$$= k2^{k+1} + 1,$$

and so $RM^{(h)}(k) = m(\log_2 m - 1) + 1$, as required.

Consider the vectors formed in the proof of Lemma 16 that count range matches. Each position in the righthand vector $(1, 0)$ is clearly largest possible over all selections of ranges, as are the preceding positions $(2, 2)$. Similarly, none of the values in $(3, 3, 3, 3)$ can possibly be greater than 3: in each case the three matches result from inequalities in the last positions of the ranges being matched. We see that in fact the vector corresponding to $x_k$ must be maximal, and so, when each range match requires constant time (proportional to $\sigma$):

**Lemma 17** Algorithm PNSV$^*$ computes $\lambda_x$ in $O(n \log n)$ time for all $x$.

Consider now the execution of NPNSV$^*$ on the strings $x_k$. Instead of one comparison per range match by PNSV$^*$, now $h+1$ letter comparisons are required. For $h = 1$, the number of comparisons per range match is therefore 2, a multiple by a constant factor, thus still linear time per match. For arbitrary $h > 2$, the number of comparisons increases by a factor of $h$, but at the same time range length (and therefore string length) increases by a factor of $(h+1)/2$, so that still $O(n \log n)$ ranges are processed in $O(n \log n)$ time. Thus

**Lemma 18** Algorithm NPNSV$^*$ computes $\lambda_x$ in $O(n \log n)$ time for all $x$. 