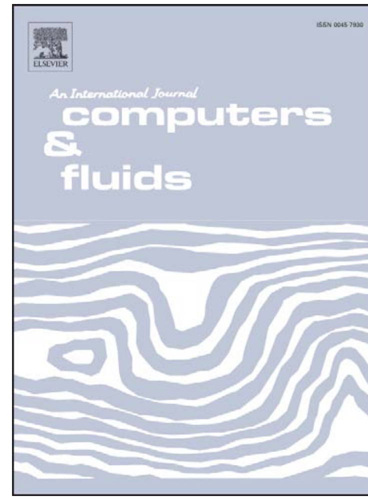

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Withdrawal from the lens of freshwater in a tropical island: the two interface case.

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Abstract

Fresh water held in the soil beneath a tropical island is one source of drinking water for the island population. If recharge through rainfall is insufficient, this resource may drain away. This work considers the circumstances under which artificial recharge will maintain the lens of freshwater. A Green function approach is used to derive an integral equation that is solved numerically for the case in which there exist two interfaces - one between salt and freshwater and one between freshwater and air. There appear to be bounds on the flow rates that produce steady interface shapes, but the height of the seepage faces is affected much more by the density ratios than the flow rates. Several different scenarios of withdrawal and influx are considered with a goal of determining some optimal management strategies.

1 Introduction

The soil beneath a tropical island may contain a lens of fresh water floating on a salt water layer; see Figure 1. Langevin et al [5] and Ruppel et al [9] discuss the importance of the management of this resource by communities

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living on small islands. The continued existence of this natural, potable water supply is governed by recharge through rainfall, the location of the interface between the fresh and saltwater layers and the volume of withdrawal. If there is insufficient rainfall, or if too much water is extracted, the lens may diminish or disappear. To prevent this it is necessary to replenish the water supply artificially. The underground storage of water has the advantage of eliminating evaporation, but the possible disadvantage of water flowing out into the ocean through the seepage face.

Chen and Hocking [1] considered circumstances under which artificial recharge could be used to maintain the water level and found that provided there was some inflow and no withdrawal, steady interfaces could be maintained. This work followed on from that of Hocking and Forbes [3], and Forbes et al [2], who computed flows due to withdrawal from the freshwater layer in the two-dimensional and axisymmetric cases assuming that the soil was fully saturated within the island. That work includes an implicit assumption that there is sufficient recharge to maintain the water levels no matter how much is withdrawn, and only involved a single interface at the freshwater-saltwater boundary. The computation of the critical coning behaviour of the interface is analogous to similar problems in unbounded domains, e.g. [4, 6, 7, 10]. However, much of that work was based in unbounded domains and so there was no necessity to balance inflow and withdrawal to obtain steady flows.

In this paper, Green's second identity and an appropriate Green's function are used to derive a boundary integral equation for the unknown location of the two interfaces. The output generated gives the shape of the two interfaces and the height of the seepage faces $\beta_\pm$ after incorporating the relevant parameters, such as density ratios γ_1 and γ_2, inflow rate, μ, and island length, α.

We consider the case where there is no natural rainfall to recharge the aquifer and artificial pumping is required to recharge the layer of fresh water. We allow the presence of an upper interface between the fresh water and the air. The work extends that of Chen and Hocking [1], in which there was no withdrawal, to consider the situation in which water is also being withdrawn, simultaneously to the recharge. Several different configurations of source and sink location are tested to determine the behaviour of the system. In [1], the inflowing water was balanced by outflow through the seepage face. The seepage faces turned out to be quite small in elevation and are affected little by changes in the flow field. In the current work we show that given a certain influx rate, there is a limit on withdrawal which corresponds to the event of coning of either the top or bottom interfaces or to the withdrawal rate becoming comparable to the inflow, thereby creating an unsustainable situation. The inclusion of direct withdrawal can be used to minimize losses through the seepage faces.

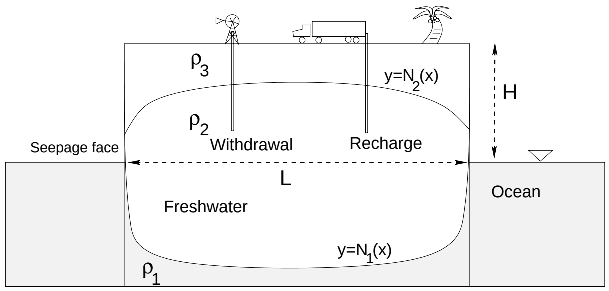

Figure 1: Sketch of tropical island of width L and height H with recharge via a line source and withdrawal through a line sink.

Section 2 derives the equations of the problem. Unlike the single free surface case, there is no recharge of fluid through the upper surface of the island. In Section 3, we formulate the problem as an integral equation using a Green function. The details of the numerical scheme are given in Section 4 and results and conclusions follow.

2 Problem formulation

Consider three fluids of different density: air, freshwater and saltwater in a porous medium of length, L, and height, H, above sea level. We choose a Cartesian coordinate system (x, y) centred in the middle of the island at sea level. The seepage velocity vector $\mathbf{q}_i$ in each layer $i = 1, 2, 3$ is given by Darcy's Law as

$$\mathbf{q}_i = -\kappa \nabla (p_i + \rho_i g y) \tag{1}$$

where κ is the total permeability of the rock and p_i is the pressure. We define the piezometric heads as

$$\Phi_i = p_i + \rho_i g y, \quad i = 1, 2, 3 \tag{2}$$

where the subscripts $1, 2, 3$ denote the variables corresponding to the layers of saltwater, freshwater and air, respectively. Assuming that the rock is fully saturated means that the continuity equation is

$$\nabla \cdot \mathbf{q}_i = 0 \quad \text{ for } i = 1, 2, 3. \tag{3}$$

Noting equation (1) and assuming the value of κ to be constant leads to Laplace's equation

$$\nabla^2 \Phi_i = 0, \quad i = 1, 2, 3 \tag{4}$$

in each of the three layers. Here, the majority of the flow occurs within the freshwater layer, and as a consequence the flow in the air and saltwater layers is assumed to be negligible so that Φ_1 and Φ_3 are constant. The region of fresh water is then bounded by the lower interface, $y = N_1(x)$, between salt and freshwater and the upper interface, $y = N_2(x)$, between air and fresh water. Across the two interfaces, the pressures must match, so that

$$\Phi_2 = \Phi_1 - (\rho_1 - \rho_2)N_1 g \quad \text{on } y = N_1(x) \tag{5}$$
$$\Phi_2 = \Phi_3 - (\rho_3 - \rho_2)N_2 g \quad \text{on } y = N_2(x), \tag{6}$$

and since there is no flow through the two interfaces,

$$\mathbf{q}_2 \cdot \mathbf{n}_k = 0 \quad \text{on } y = N_k(x), \quad k = 1, 2 \tag{7}$$

where $\mathbf{n}_k$ are the normals to the interfaces, $k = 1, 2$.

The two seepage faces are in contact with the air, and therefore

$$\Phi_2 = -(\rho_3 - \rho_2)gy \quad \text{on } x = \pm L, \quad 0 < y < B_\pm \tag{8}$$

where $B_\pm$ refers to the heights of the seepage faces at $x = \pm L/2$, respectively. Non-dimensionalizing with respect to the island height H, and the middle layer potential $\rho_2 g H$, and letting $\rho_1/\rho_2 = \gamma_1 > 1$ and $\rho_3/\rho_2 = \gamma_2 < 1$, gives

$$\phi_2 = \phi_1 + (1 - \gamma_1)\eta_1 \quad \text{on } y = \eta_1(x), \tag{9}$$
$$\phi_2 = (1 - \gamma_2)\eta_2 \quad \text{on } y = \eta_2(x), \tag{10}$$
$$\phi_2 = y \quad \text{on } x = \pm\alpha/2, \quad 0 < y < \beta_\pm. \tag{11}$$

for the potential ϕ_2 and interfaces $\eta_1(x)$ and $\eta_2(x)$ on an island of nondimensional width $\alpha = L/H$ with seepage faces of height $\beta_\pm = B_\pm/H$ at $x = \pm\alpha/2$. Without loss of generality we have set $\phi_3 = 0$ and ϕ_1 is constant. Henceforth, we will drop the subscript '2' for the middle layer potential.

Finally, the existence of a line source or sink at any point (x_s, y_s), requires that

$$\phi(x, y) \to \frac{\mu}{2\pi} \ln \left[(x - x_s)^2 + (y - y_s)^2\right]^{1/2} \quad \text{as } (x, y) \to (x_s, y_s). \tag{12}$$

Note that $\mu < 0$ corresponds to a source flow. In this paper we will include several sources and sinks at different locations, at which this property must be satisfied.

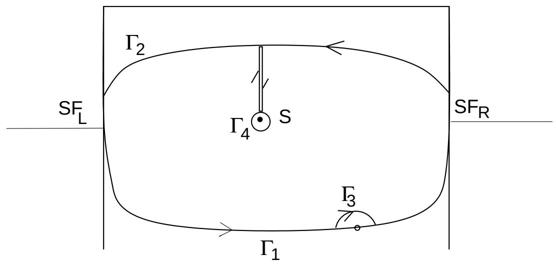

Figure 2: Schematic diagram of island of length α and the integration paths Γ_1, Γ_2, Γ_3, Γ_4 and the seepage faces SF_L and SF_R. Γ_3 goes around the point (x_0, y_0), while Γ_4 around the source point (x_s, y_s). If there are several sources/sinks in the domain then this Γ_4 loop must be repeated for each.

3 Integral equation for two free surfaces

A Green function approach similar to that used by Hocking and Forbes [3] and Chen and Hocking [1] is used. We seek a function G that satisfies (13) subject to the condition (14), where

$$\nabla^2 G = \delta(x - x_0, y - y_0) \tag{13}$$

$$\text{with } G(\pm\frac{\alpha}{2}, y; x_0, y_0) = 0; \qquad -\infty < y < \infty \tag{14}$$

where $\delta(x - x_0, y - y_0)$ is the Dirac-delta function.

These conditions set $G = 0$ at $x = \pm\frac{\alpha}{2}$, along the seepage faces. Using conformal mapping techniques, a suitable form for G is

$$G = \frac{1}{4\pi}\ln[(f - f_0)^2 + (g - g_0)^2] - \frac{1}{4\pi}\ln[(f - f_0)^2 + (g + g_0)^2] \tag{15}$$

and we determine that

$$G_x = \frac{1}{2\alpha}\left[\frac{(f - f_0)g - (g - g_0)f}{(f - f_0^2 + (g - g_0)^2} - \frac{(f - f_0)g - (g + g_0)f}{(f - f_0)^2 + (g + g_0)^2}\right] \tag{16}$$

$$G_y = \frac{1}{2\alpha}\left[\frac{(f - f_0)f + (g + g_0)g}{(f - f_0)^2 + (g + g_0)^2} - \frac{(f - f_0)f + (g - g_0)g}{(f - f_0)^2 + (g - g_0)^2}\right] \tag{17}$$

where $f = f(x, y)$, $g = g(x, y)$, $f_0 = f(x_0, y_0)$ and $g_0 = g(x_0, y_0)$, and

$$
\begin{aligned}
f(x,y) &= e^{-\pi y/\alpha} \sin \frac{\pi x}{\alpha} && (18)\\
g(x,y) &= e^{-\pi y/\alpha} \cos \frac{\pi x}{\alpha} && (19)\\
f_x(x,y) &= \frac{\pi}{\alpha} e^{-\pi y/\alpha} \cos \frac{\pi x}{\alpha} = \frac{\pi}{\alpha} g && (20)\\
f_y(x,y) &= -\frac{\pi}{\alpha} e^{-\pi y/\alpha} \sin \frac{\pi x}{\alpha} = -\frac{\pi}{\alpha} f && (21)\\
g_x(x,y) &= -\frac{\pi}{\alpha} e^{-\pi y/\alpha} \sin \frac{\pi x}{\alpha} = -\frac{\pi}{\alpha} f && (22)\\
g_y(x,y) &= -\frac{\pi}{\alpha} e^{-\pi y/\alpha} \cos \frac{\pi x}{\alpha} = -\frac{\pi}{\alpha} g && (23)
\end{aligned}
$$

Now consider Green's second identity

$$
\iint_A (\phi \nabla^2 G - G \nabla^2 \phi)\, dA = \int_\Gamma \left(\phi \frac{\partial G}{\partial \mathbf{n}} - G \frac{\partial \phi}{\partial \mathbf{n}} \right) dS \qquad (24)
$$

where A refers to the interior of the domain bounded by the upper and lower interfaces and the two seepage faces, denoted as Γ; see Fig. 2. Noting that $\nabla^2 \phi = 0$ everywhere except at (x_s, y_s), the source location, and that G satisfies Laplace's equation, that is $\nabla^2 G = 0$ except at (x_0, y_0), and that $G = 0$ along the seepage face, the left hand side of the identity is zero if the path of integration omits (x_0, y_0) and (x_s, y_s). Therefore, if we choose the path of integration to be around the boundary but with small circles that exclude the singularities at (x_0, y_0) and (x_s, y_s) as shown in Figure 2, we are left with an integral equation for integration around the boundary of region 2 only.

After careful substitution of the boundary conditions where $\frac{\partial \phi}{\partial \mathbf{n}} = 0$ and $G = 0$ as appropriate, adding and subtracting a term to remove the singularity as $(x, y) \to (x_0, y_0)$, and carefully integrating around the loops Γ_3 and Γ_4, we are left with

$$
\int_{-\alpha/2}^{\alpha/2} (\phi - \phi_0)[\eta_1' G_x - G_y]\, dx + \int_{-\alpha/2}^{\alpha/2} (\phi - \phi_0)[\eta_2' G_x - G_y]\, dx
+ \sum_{k=1}^{NS} \frac{\mu_k}{4\pi} \ln \left[\frac{(f_{s_k} - f_0)^2 + (g_{s_k} - g_0)^2}{(f_{s_k} - f_0)^2 + (g_{s_k} + g_0)^2} \right] + I_1 + I_2 = 0 \qquad (25)
$$

where NS is the number of sources/sinks in the flow domain, μ_k, and (x_k, y_k), $k = 1, 2, \ldots, NS$ are their respective strengths and locations, and I_1 and I_2

correspond to integrals along the seepage faces,

$$
\begin{aligned}
I_1 &= \frac{1}{\alpha}\int_0^{\beta_+}(\phi-\phi_0)\left[\frac{e^{-\pi y/\alpha}(e^{-\pi y_0/\alpha}\cos\frac{\pi x_0}{\alpha})}{(e^{-\pi y/\alpha}-e^{-\pi y_0/\alpha}\sin\frac{\pi x_0}{\alpha})^2+(e^{-\pi y_0/\alpha}\cos\frac{\pi x_0}{\alpha})^2}\right]dy\\
I_2 &= \frac{1}{\alpha}\int_0^{\beta_-}(\phi-\phi_0)\left[\frac{e^{-\pi y/\alpha}(e^{-\pi y_0/\alpha}\cos\frac{\pi x_0}{\alpha})}{(e^{-\pi y/\alpha}+e^{-\pi y_0/\alpha}\sin\frac{\pi x_0}{\alpha})^2+(e^{-\pi y_0/\alpha}\cos\frac{\pi x_0}{\alpha})^2}\right]dy
\end{aligned}
\tag{26}
$$

and $\phi_0 = \phi(x_0, y_0)$ which, combined with (9), (10) provides a closed system for the unknown locations of the two interfaces, $\eta_1(x), \eta_2(x)$. The sources and sinks can be placed anywhere within the fluid domain.

4 Numerical method

The integral equation (25) is highly nonlinear and therefore we adopt a numerical approach. A discrete approximation is found by taking boundary points $x_j = x_1, x_2, \ldots, x_n$ in x with the aim to find ϕ_j, η_{1_j} and η_{2_j} for $j = 1, 2, \ldots, n$. This gives rise to $2n$ equations in $2n$ unknowns, each corresponding to a point in the discrete representation of $-\alpha/2 < x < \alpha/2$. The discretised equation is

$$
\begin{aligned}
&\sum_{j=0}^{n}(\phi_j-\phi_i)[\eta_1'(x_j)G_{x_j}-G_{y_j}]\Delta x_j w_j\\
&\quad+\sum_{j=0}^{n}(\phi_{n+j}-\phi_i)[\eta_2'(x_j)G_{x_j}-G_{y_j}]\Delta x_j w_j\\
&\qquad+\sum_{k=1}^{NS}\frac{\mu_k}{4\pi}\ln\left[\frac{(f_{s_k}-f_i)^2+(g_{s_k}-g_i)^2}{(f_{s_k}-f_i)^2+(g_{s_k}+g_i)^2}\right]+(I_1+I_2)\\
&\quad=0, \qquad \text{for } i=1,2,\ldots,2n
\end{aligned}
\tag{27}
$$

where $\Delta x_j = x_{j+1} - x_j$ is the step size and w_j the weighting for the trapezoidal rule. I_1 and I_2 are as in equations (26) and can be accurately evaluated using standard techniques.

The values of ϕ_i correspond to those on both surfaces, and can be replaced using conditions (9) and (10), leaving the unknown interface locations η_{1i} and η_{2i}, $i = 1, 2, \ldots, n$ as the $2n$-unknowns. In fact, the end points of the lower interface are known to lie at $(-\alpha/2, 0)$ and $(\alpha/2, 0)$ and so these are omitted from the scheme, leaving $2n - 2$ nonlinear equations in $2n - 2$ unknowns.

In (25) there is a possible singularity in the integrand as $(x, \eta) \rightarrow (x_0, \eta_0)$. Carefully taking the limit in the two integrals as $(x, y) \rightarrow (x_0, y_0)$ gives

$$\lim_{x \rightarrow x_0} (\phi - \phi_0)[\eta_k'(x) G_x - G_y] = \frac{1}{2\pi}(1 - \gamma_k)[\eta_k'(x)]^2, \quad k = 1, 2. \tag{28}$$

These values were incorporated into the trapezoidal integration scheme.

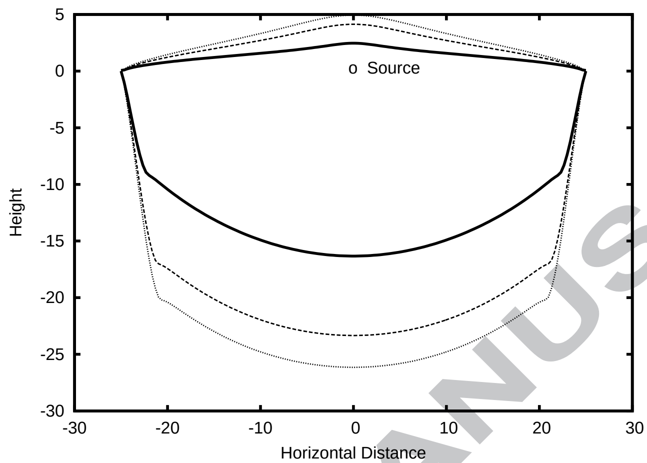

Figure 3: Typical interface shapes for an island of width $\alpha = 50$ with $\mu_1 = -1.65$ (solid), $\mu_1 = -4$ (dash), $\mu_1 = -5.6$ (dots) for density ratios $\gamma_1 = 1.1$ and $\gamma_2 = 0$. Note that negative μ values indicate a source flow.

A damped Newton's method was used to solve the system of $2n - 2$ nonlinear equations. Thus, an initial guess for the values of $\eta_1(x_i)$, $i = 2, \ldots n-1$ and $\eta_2(x_i)$, $i = 1, 2, \ldots, n$, was updated iteratively. Most simulations were performed with a space step of $\Delta x = 0.1$ and this gave graphical accuracy for the interfaces and seepage face heights. In circumstances where one of the interfaces was approaching a maximal coning configuration, more accurate simulations (larger values of n) were required to obtain converged solutions.

5 Results

Simulations were carried out for a range of different inflow rates with $\gamma_1 = 1.5$ and $\gamma_1 = 1.1$, both with $\gamma_2 = 0$ (air-water) and for many different island lengths and source and sink locations. For given values of μ and γ_1, there is a corresponding seepage face height β to which the numerical scheme will converge. Chen and Hocking [1] computed a number of solutions in which withdrawal was not included in order to determine some baseline behaviour of the aquifer during recharge.

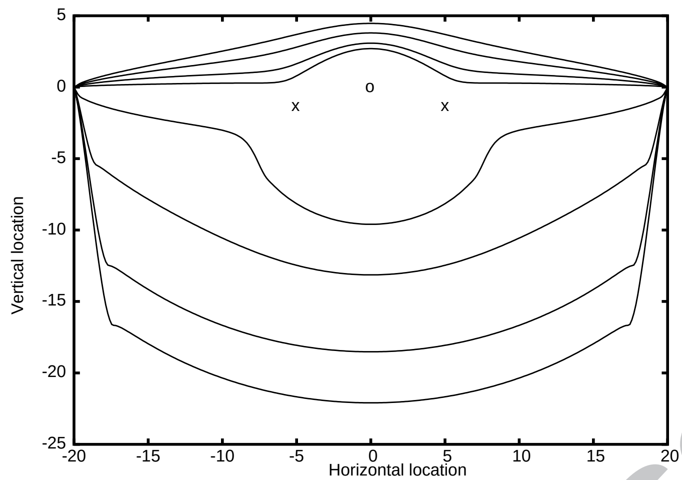

Figure 4: Interface shapes for an island of width $\alpha = 40$ with a source ($\mu_1 = -6.0$) at $(0, 0)$ shown as 'o' and with sinks at $(\pm 5, -1.5)$, shown as 'x'. Sink strengths are $\mu_2 = \mu_3 = 0.5, 1.5, 2.5, 2.95$ (maximal). Lowest strengths correspond to lines furthest from the source/sinks. Coning is evident on both interfaces in the maximal flow, but is not critical.

Typical results are given for the zero withdrawal case in Figure 3, which shows the interface shapes for the case of an island of length $\alpha = 50$ where the recharge source is located at $(x_s, y_s) = (0, 0)$ with $\gamma_1 = 1.1$ and $\gamma_2 = 0$. As the pumping rate μ increases, the height (and depth) of the two interfaces increases almost linearly. Notice that the seepage face is very small and the depth of the lower interface is much greater. This agrees with the results of Polubarinova-Kochina, [8], who used approximate methods to show that $\eta_1(x) \propto 1/(1 - \gamma_1)$. The actual value of the depth of the lower interface is determined by the nett recharge rate.

In this paper, however, our main concern is the results of a series of simulations with different source and sink locations. Although it is possible to work with many such points, we generally restrict simulations to only one or two of each type of singularity. Results indicate that depending on the locations and recharge rates there is a maximum withdrawal that can be obtained for any configuration. If coning occurs, this provides the maximum withdrawal flow, but if the withdrawal reaches a magnitude close to the total recharge then the maximal flow is due to this limit. The nett flow must be into the aquifer or there can be no steady solution. It does appear to be possible to "mask" the sink or sinks from the interface by judicious placement of the inflow sources, thus allowing higher withdrawals. Several sequences of simulations have been conducted to illustrate these points.

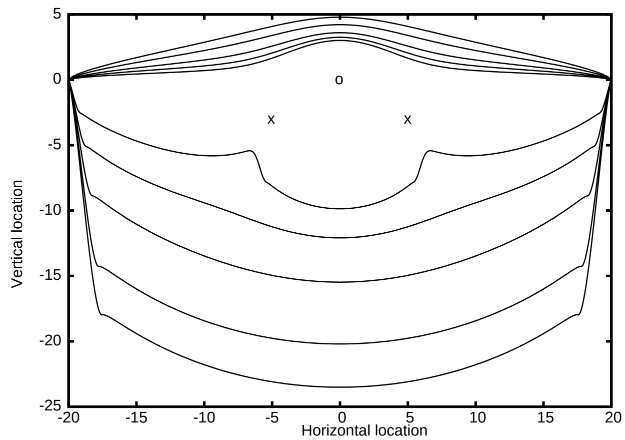

Figure 5: Interface shapes for an island of width $\alpha = 40$ with a source ($\mu_1 = -6.0$) at $(0, 0)$ shown as 'o' and with sinks at $(\pm 5, -3.5)$, shown as 'x'. Sink strengths are $\mu_2 = \mu_3 = 0, 1, 2, 2.5, 2.78$ (maximal). Lowest strengths correspond to lines furthest from the source/sinks. The lower interface is approaching the critical withdrawal cone.

In the first sequence of simulations a source of strength $\mu_1 = -6.0$ (negative values indicate source flow) was placed at the origin $(0, 0)$ and two sinks either side at $x_s = \pm 5$ were moved vertically to determine the outcome. Thus, several cases were considered with sinks at $(\pm 5, y_s)$ where y_s took different values. The solutions indicate that the nearest interface is the first to begin to be drawn toward the outlets and if one of the two interfaces is sufficiently close then coning may occur before any significant local deformation to the other occurs.

Figure 4 shows the case with sinks at $(\pm 5, -1.5)$, in which both interfaces show signs of being drawn toward the outlets. The maximum value in this case is due to the requirement that the nett recharge must be positive. Neither interface was close to critical coning when the maximum solution was obtained. The total flow rate due to the two sinks with $\mu_2 = \mu_3 = 2.95$ is therefore 5.9, just below the influx of 6.0.

In contrast, Figure 5 shows the case with a single source and two sinks located at $(\pm 5, -3.5)$ as the strengths of the two sinks are increased through values of $\mu_2 = \mu_3 = 0, 1, 2, 2.5, 2.78$ (close to the maximum withdrawal). The nearby lower interface has been pulled up sharply into a cone while the effect on the upper interface is an island-wide drop in level with only a small local bump due to the withdrawal. The maximal flow here is most likely due to critical coning behaviour of the lower interface.

Figure 6 shows the maximum total withdrawal as a function of sink depth

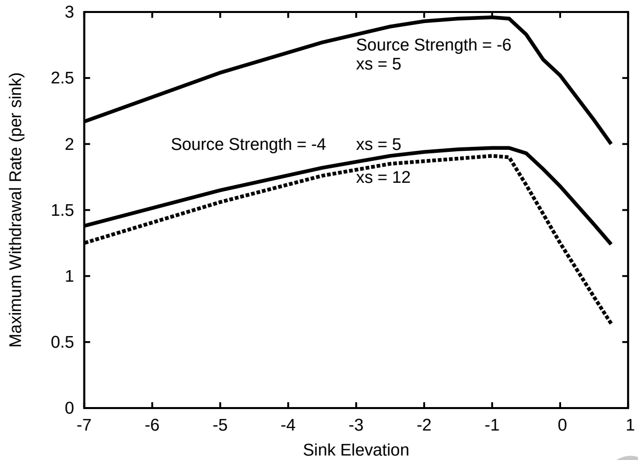

Figure 6: Maximum withdrawal (per sink) as a function of sink elevation for different source strength (the two solid lines) and one different horizontal location (dashed line).

and it is clear that over the range $y_s \in (-2.5, -1)$ the limit is due to the requirement of a positive discharge rather than a critical coning limit (since the outflow flux is close to the limit). It is also clear that as the sink moves upward above $y_s = -1$ the upper surface is close enough to the sink for the coning limit to apply, especially for the case when the sinks are moved outward away from the source and the "shielding" of the interface by the source is diminished.

The second sequence concerns the effect of horizontal location of the sinks. Figure 7 shows the maximal flow for each of the cases with sources of strength $\mu_1 = -3, -4, -6$ at $(0, -2)$ and the sinks at $(\pm 1, 0), (\pm 5, 0), (\pm 10, 0), (\pm 15, 0)$ and $(\pm 17.5, 0)$. In all of these cases the maximum corresponds to critical coning due to the proximity of the upper interface. This could be avoided by placing the sinks deeper underground, but the point here is that closer to the island edge the maximum flow is reduced by the proximity of the interface and the edge of the island, which induces a kind of "ground effect", increasing the apparent sink strength.

In order to consider a slightly different configuration, a third sequence of simulations was performed in which the withdrawal point was centred at $(x_s, y_s) = (0, -2)$ and the sources were placed at varying horizontal locations. This is a reversal in sign of the second sequence. Total source strength was again fixed and sink strength gradually increased. Figure 8 shows a plot of interface heights for two sources of total strength $\mu_2 + \mu_3 = -6$ located at $(\pm 1, 0)$. The shielding effect of the sources on the sink is clearly seen on the

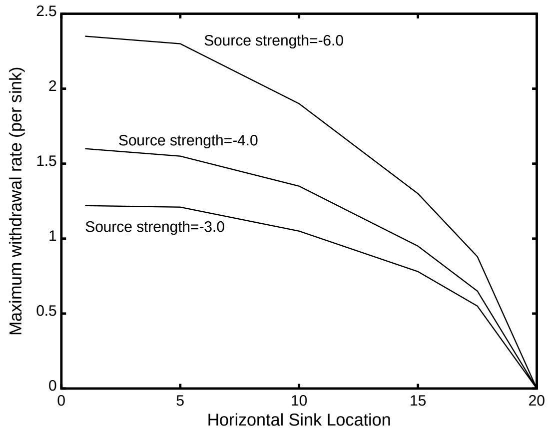

Figure 7: Maximum withdrawal rates for an island of width $\alpha = 40$ and $\gamma = 1.1$ with sources of strength $\mu_1 = -6.0, -4.0, -3.0$ at $(0, -2)$ and sink pairs at $(\pm x_s, 0)$ where $x_s = 1, 5, 10, 15, 17.5$.

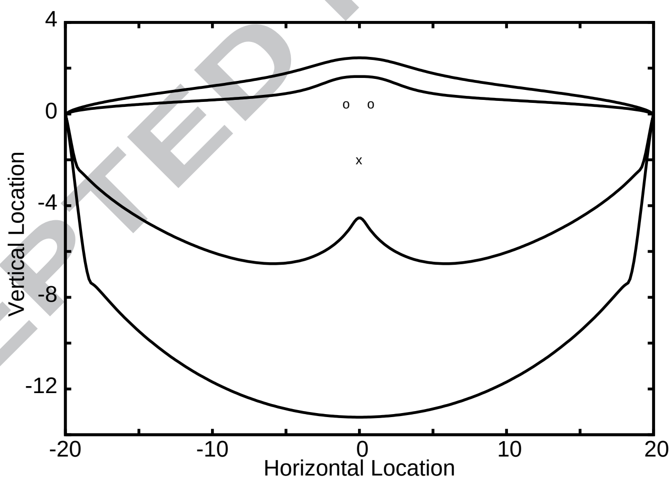

Figure 8: Interface shapes for an island of length $\alpha = 40$ and sources of strength $\mu_2 = \mu_3 = -3$ at $(-1, 0), (1, 0)$, shown as 'o', and a sink, 'x', at $(0, -2)$ where $\mu_1 = 2.6, 3.6$. Shielding of the sink by the sources is evident for the top interface.

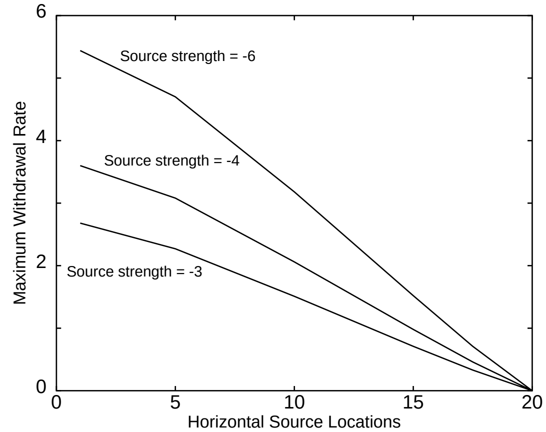

Figure 9: Maximum withdrawal rates for an island of width $\alpha = 40$ and $\gamma = 1.1$ with a single sink at $(0, -2)$ and source pairs with total strength $\mu_2 + \mu_3 = -6.0, -4.0, -3.0$ at $(\pm x_s, 0)$ where $x_s = 1, 5, 10, 15, 17.5$.

top interface as the sources push the interface upward even though the source is in close proximity. The lower interface is beginning to cone, however.

The results for a range of this reverse flow case are given in Figure 9 in which the maximal withdrawal is now plotted against the source locations. In almost all cases it is the maximum flux condition that limits the flow rather than the coning of the interface. This suggests that having the sources outside of the sinks may provide a better solution in which less water drains into the ocean through the seepage face.

Finally, all of the above configurations were symmetric as it would seem most likely that the most efficient withdrawal patterns would be occur in this way since if an optimal sink location at one end of the island were found, it seems likely the same set up would be optimal at the other. However, this assumption is not necessary to compute solutions and Figure 10 shows one such solution. It is clear that the sink, located at $x_s = -10$, has significantly drawn down the interface locally while having almost no effect on the far end of the island.

6 Concluding Remarks

We have developed a method to solve for the interfaces of the lens of freshwater beneath an island. No recharge is allowed through the surface of the lens, and artificial recharge of water is through some number of line sources. Withdrawal through line sinks is included to consider the likelihood of coning

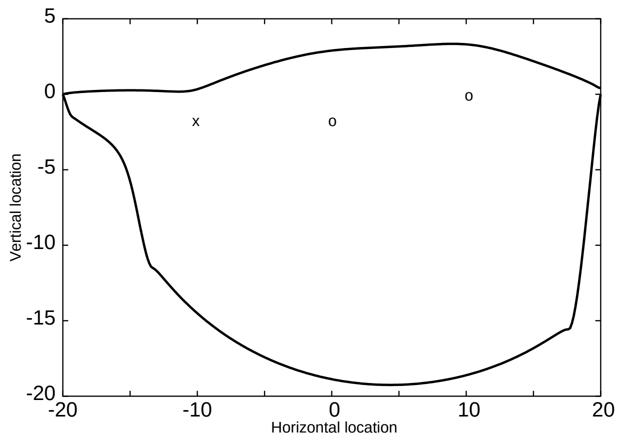

Figure 10: Interface shapes at maximum withdrawal rates for an island of width $\alpha = 40$ and $\gamma = 1.1$ a non-symmetric distribution of sources at $(0, -2)$ and $(10, 0)$ with total strength $\mu_1 + \mu_2 = -6.0$ and a sink located at $(-10, -2)$. There has been significant draw up of the left end due to the withdrawal at this end of the island.

in different situations. In order to obtain a steady state, seepage faces are allowed at both ends of the island and the total inflow must be greater than withdrawal. Solutions have been obtained over a range of density differences, source and sink strengths and locations and island lengths.

It was shown in Chen and Hocking [1] that the magnitude of the saturated zone above sea level could be quite high, so that it would seem likely that the upper interface would reach to the surface of the island in many cases, giving the situation considered by Hocking and Forbes [3]. No limit was placed on the island height in the current work.

The work differs from much of the previous work by allowing a second interface. It shows that a steady, stable lens can be maintained using a single or multiple recharge points, and the resulting seepage face heights are almost totally dominated by the density ratios between the fluid layers. Withdrawal can be conducted up to close to the same value as the recharge so long as the outlets are placed appropriately. Coning can be prevented if the outlets are placed close to the influx point. However, it is likely in this situation that much of the outflow is coming almost immediately from the inflow source and so any positive filtration effects will be diminished. This shielding effect could be utilized by placing more, lower strength sources and sinks (albeit at greater cost) in close proximity to each other along the island, allowing almost everything that is pumped in to be removed without too much loss to the ocean.

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Research Highlights for CAF1645

"Withdrawal from the lens of freshwater in a tropical island: the two interface case"

- Withdrawal from a lens of fresh water in porous media under an island is considered
- Deformations of interfaces are considered with both withdrawal and recharge
- Depending on placement of inflow and withdrawal points, small losses can be managed
- Inflow points can be used to shield the interface from drawdown into the outlets
- Critical drawdown flow rates are obtained for a range of situations