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Estimating the Robust Domain of Attraction with Bounded Uncertainties via Markov Models

†Reihane Kardehi Moghaddam, ‡Mehdi Kardehi Moghaddam

Abstract— **In this work a new approach for estimating the robust domain of attraction of dynamical systems with bounded uncertainty is proposed. We analysis the stability of dynamical systems by Markov modeling which focuses on asymptotic behaviors of systems. The proposed method expresses the problem of estimating robust domain of attraction as an infinite dimensional linear problem. Using approximated Markov transition function, the resulting linear problem is converted to a finite dimensional optimization problem. The efficiency of proposed methods is shown via simulations.**

Index Terms— **Discrete dynamical systems, domain of attraction, invariant measure, Perron-Frobenius theorem, Markov chain, robust domain of attraction.**

I. Introduction

In this paper we propose a new method for estimating Robust Domain of Attraction (RDA) using Markov modeling. Some advantages of using Markov models for extracting dynamical behaviors is that the statistical properties of this model often are easily computed numerically and using Markov chains, one can just compute the asymptotic behaviors of system which takes less time than direct analysis of system orbits. Any analysis of dynamical systems involving average quantities requires a reference measure to average contributions from different regions of the phase space. The most popular measure used in these cases is the probability invariant measure, which is described by the distribution of the typical long trajectories of the system [1].

As most of physical systems have uncertain parameters, finding RDA that guarantees the stability for different values of uncertainty is very important and of most interest. Although calculating actual RDA remains unsolved problem, some solutions for estimating this region are suggested in recent literatures such as finding a common Lyapunov function (LF) to prove robust local stability [2], estimating RDA via parameter dependant LF [2, 3] and RDA estimation through generalized Zubove's method.

Above methods have some limitations in estimating RDA. Although RDA of systems with probably time varying uncertainty can be estimated through common LF, finding such a common LF in general is impossible. Parameter dependant LF is applicable only for time invariant uncertainties. In addition, there is not a general LF structure and most literatures use quadratic LF which leads to a conservative estimation of DA. In the third method, the viscosity solution of straightforward generalization of classical Zubove's equation is used to characterize RDA of a nonlinear system with time varying perturbations [4]. Zubove's method is concerned with exact determination of DA [5] and has some limitations. For example, to solve the Zubove's equation, method of characteristic is used but this method requires solution of nonlinear system and in fact the knowledge of DA which is mostly impossible. Our work overcomes these limitations using invariant measure as an approximating tool. Although the performed finite state model in this work has less information than the original system, this simplification allows computing some dynamical properties such as finding invariant sets and estimating RDA for a large class of nonlinear systems effectively.

This work contains 4 sections. In the second section some definitions are summarized. Introducing the stability theorems according to Markov model and RDA estimation by means of Markov chains and invariant measure are subjects of third section. And finally, in the section four the results are simulated.

II. Definitions

Let Ω be an n-dimensional open rectangular set in R^n, equipped with Lebesgue measure μ on σ-algebra of Borel sets $B(\Omega)$ and T be a measurable nonsingular transition operator [1] on the measurable space (Ω, B, μ) such that

$$X(k+1) = T(X(k)) \qquad T:\Omega \to \Omega,\ \Omega \subset R^n$$
$$X(k) = [x_1(k),..., x_n(k)]^T, x_i(k+1) = T_i(X(k)) \tag{1}$$

For the above system the following definitions are considered:

Definition 1: State space partitioning

$\mathcal{A}$ is a state space partitioning for Ω if it divides Ω into cells A_i $i = 1,..., N$ such that they satisfy the following two conditions:

$$\bigcup_{i=1}^{N} A_i = \Omega \tag{2}$$
$$\overset{\circ}{A}_i \cap \overset{\circ}{A}_j = \phi \ \forall i \neq j$$

Where $\overset{\circ}{A}_i$ is the interior of A_i set and ϕ is the empty set.

Definition 2: Center of a partition

Let $\mathcal{A}$ be a state space partitioning for $\Omega \subset R^n$. For simplicity we suppose rectangular partitions as

Reihane .K. Moghaddam, Department of Electrical engineering, Mashhad Branch, Islamic Azad University, Mashhad, Iran.
(corresponding author phone: +98 915 101 6955;e-mail: rkardehi.moghaddam@gmail.com,R_kardehi_moghaddam@mshdiau.ac.ir)

Mehdi. K. Moghaddam, School of Engineering and Energy, Murdoch University, Western Australia, 6150 (e-mail: M.Kardehimoghaddam@murdoch.edu.au)

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$A_i = [l_{1i}, h_{1i}] \times \ldots \times [l_{ni}, h_{ni}]$ $i = 1,.., N$. The center of each partition A_i is a point like $C_i = [c_{1i}, ..., c_{ni}]^T$, where $c_{ji} = \dfrac{h_{ji} - l_{ji}}{2}$.

Definition 3: Robust domain of attraction
Consider an uncertain nonlinear system of the following form:

$$
\begin{aligned}
&X(k+1) = T_U(X(k), \beta) \\
&T_U(X_e, \beta) = X_e \quad \forall \beta \in B \\
&T_U : \Omega \times B \to \Omega, \ \Omega \subset R^n, \ B \subset R^p
\end{aligned}
\qquad (3)
$$

where $\beta \in B$ is an uncertainty vector and B is a measurable compact set in R^p and T_U is a nonsingular uncertain transition operator. The robust domain of attraction of system (3) is defined as $RDA = \{X(k) \in \Omega | \ \lim_{h \to \infty} T_U{}^h (X(k), \beta) = X_e ; \forall \beta \in B\}$

Obviously $T_U(X_e, \beta) = X_e \ \ \forall \beta \in B$ implies that in this paper a class of nonlinear systems is considered which has at least one isolated equilibrium point that is not sensitive to parameters variations.

Definition 4:
The Markov chain for discrete system (1) is as follows [7]:

$$\phi = \left\{ \phi_k \middle| \ \ \phi_k = X(k) = T^k(X(0)), \ 0 \le k < n+1 \right\} \quad (4)$$

Definition 5:
Let $X \in \Omega$ and $A \subset \Omega$. The n-step transition function, denoted by $p^n(X, A)$, shows the probability that a Markov chain ϕ starting from an arbitrary point like $\phi_0 = X$ remaining in the set A after n steps [6].

III. ESTIMATING RDA

A. Stability theorems and definitions

As we are concerned with estimating domain of attraction of uncertain (or certain) systems, we should analyze the long term orbits of the system, but it is not practically possible in many systems because it takes a long time and may lead to computer round off error. Therefore, in this paper we use the method of Markov modeling of dynamical systems to remove the transient effects and calculate the asymptotic behaviors.

Proposition 1:
For Markov chain (4) the Markov transition function is proposed as $P(X, A) = \lim_{n \to \infty} p^n(X, A)$.

Proof: see [6, chapter 1, page 3].

Theorem 1:
The existence of a fixed point like X_e which is asymptotically stable in the set $A \subset \Omega$ is exactly equal to the existence of a nonzero unique solution for the following invariant equation:

$$m(A) = \int_\Omega P(X, A) \ dm(A)$$

Proof: see [6, chapter 1, page 20, asymptotic stability definition].

In the above theorem $m \in M$ and M is the set of all probability Lebesgue measures on the topological space Ω.

Lemma1:
Closure of the Domain of attraction of nonlinear system (1), $\overline{DA} \subset \Omega$, is the union of the members of support of probably measure m and obtained from following equation:
$\overline{DA} = SUPP\{m\}$
where $SUPP\{m\} = \bigcup \{A | \ m(A) = \int_\Omega P(X, A) \ dm(A) \neq 0\}$.

Proof: According to theorem1 every member of $SUPP\{m\}$ is asymptotically stable so it is contained in DA so it yields $\overline{DA} = SUPP\{m\}$. As A is a close set $SUPP\{m\}$ is also close.

It is not practically possible to estimate domain of attraction of system (1) using Lemma 1, because it leads to an infinite dimensional problem in space M. In other words since $DA \subset \Omega$, we should calculate $P(X, A) = \lim_{n \to \infty} p^n(X, A)$ for every $X \in \Omega$ which leads to an infinite dimensional problem. So we use the rough idea of [8] and partition the state space Ω (according to definition 1). Assuming that $P(X, A)$ has a uniform distribution, we can calculate probability of partitions transitions instead of calculating every point $X \in \Omega$ transition. So in the sequel we convert the infinite dimensional problem of estimating DA, proposed in Lemma1, to a finite dimensional one. To investigate the stability of state partitions we use the discrete-time Markov chain which is a Markov process ϕ_n having a countable number of states A_n [7].

Definition 6: Markov transition matrix
Consider nonlinear system (1). For $\mathcal{A}$ partitioning of Ω, the $N \times N$ Markov transition matrix P is defined as:

$$
\begin{aligned}
&P^{(n_1, n_2)} = [p_{ij}{}^{(n_1, n_2)}] = [prob\, [X(n_2) \in A_j | X(n_1) \in A_i]]; \\
&\sum_j p_{ij}{}^{(n_1, n_2)} = 1
\end{aligned}
\qquad (5)
$$

Definition 7: n-step Markov transition matrix
n-step Markov transition matrix for a homogenous Markov process is defined as:

$$P^{(n)} = [p_{ij}^{(n)}] = prob(X(k+n) \in A_j / X(k) \in A_i) \qquad (6)$$

Proposition 2: For uniformly distributed $P(X, A)$, $p_{ij}^{(1)}$ can also be presented as:

$$p_{ij}^{(1)} = \frac{m(T^{-1}(A_j) \cap A_i)}{m(A_i)} \quad i, j = 1, ..., N \qquad (7)$$

Proof: see [8].

Theorem 2:
The (closure of) domain of attraction of nonlinear system (1) with N state partitioning $\mathcal{A}$ can be estimated from the support of invariant measure vector ϑ. Where ϑ is calculated from following equations:

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$$\vartheta = P\vartheta \quad ; \vartheta = (\vartheta_1, ..., \vartheta_N)$$
$$\sum_{i=1}^{N} \vartheta_i = 1 \tag{8}$$

Proof: see [1] □

B. Proposed numerical method to find Markov transition matrix

Considering theorem 2, to estimate DA, we should calculate Markov matrix. There are different numerical algorithms to calculate P matrix from equation (7) [see chapter 6 of reference 8]. In the sequel, we provide a new analytic formula to determine P which is more accurate; moreover we use this analytic form to estimate RDA.

Proposition 3:

Some useful properties of the (probability) Lebesgue measure m and characteristic function χ are:

a- $m(A \cap B) = \int_A \chi_B(X)\,dX = \int_B \chi_A(X)\,dX$

Proof:

From [9] we have $\chi_{(A\cap B)} = \chi_{(A)} \cdot \chi_{(B)}$, which yields;

$$m(A \cap B) = \int_\Omega \chi_{A\cap B}(X)\,dX =$$
$$\int_\Omega \chi_A(X).\ \chi_B(X)\,dX = \int_A \chi_B(X)\,dX = \int_B \chi_A(X)\,dX$$

□

b- $\chi_{T^{-1}(A)}(X) = \chi_A(T(X))$

proof: Science T is nonsingular we have

$$\chi_{T^{-1}(A)}(X) = 1 \Leftrightarrow X \in T^{-1}(A) \Leftrightarrow T(X) \in A$$
$$\Leftrightarrow [T_1(X), ..., T_n(X)]^T \in A \Leftrightarrow \chi_{(A)}(T(X)) = 1$$

□

Lemma 2:

For nonlinear system (1), with state space partitioning $\mathcal{A}$, the Markov matrix can be represented by the following analytic form:

$$p_{ij} = \frac{\int_\Omega \prod_{q=1}^{n} heaviside[f_T(q)] \prod_{k=1}^{n} heaviside\,[f_x(q)]\,dX}{S_{D_j}}$$

$$where:$$
$$f_T(q) = ((T_q - l_{qj}).(h_{qj} - T_q)$$
$$f_x(q) = (x_k - l_{ki}).(h_{ki} - x_k) \tag{9}$$
$$S_{D_j} = \prod_{k=1}^{n} (h_{kj} - l_{kj}) \ , dX = dx_1 ... dx_n$$
$$heaviside(x) = \begin{cases} 1 & x > 0 \\ 0.5 & x = 0 \\ 0 & x < 0 \end{cases}$$

And the state space portioning $\mathcal{A}$ is chosen as $A_i = [l_{1i}, h_{1i}] \times ... \times [l_{ni}, h_{ni}]\ i = 1,.., N$.

Proof:

From proposition 3-a, the P matrix as defined in proposition 2 can be expressed as :

$$p_{ij} = \frac{m(T^{-1}(A_j) \cap A_i)}{m(A_i)} = \frac{\int_\Omega \chi_{T^{-1}(A_j)}(X).\ \chi_{A_i}(X)\,dX}{\int_\Omega \chi_{A_i}(X)\,dX}$$

Therefore from proposition 5-b, we have

$$p_{ij} = \frac{\int_\Omega \chi_{A_j}(T(X)).\ \chi_{A_i}(X)\,dX}{\int_\Omega \chi_{A_i}(X)\,dX} \tag{10}$$

According to characteristic function definition [7], an acceptable $\chi_{A_i}(X)$ for A_i set is:

$$heaviside \prod_{k=1}^{n} [(x_k - l_{ki}).(h_{ki} - x_k)] \tag{11}$$

Substituting (12) in (11), completes the proof. □

C. RDA estimation

In this section, we generalize the stability theorem2, defined in the previous section, for RDA estimation. Although finding the exact RDA is a difficult problem, different ways are proposed in literatures to estimate it. Some of these methods choose arbitrary values for uncertainty and estimate DA for these fixed values and finally estimate RDA from intersection of DA sets. These methods are not reliable because they just study DA variations for special values in uncertainty bound. On the other hand, as these methods are based on intersecting DAs they usually use simple Lyapunov functions for estimating DA[2]. The Lyapunov based algorithms which use quadratic structures only obtain a conservative estimate of RDA and the other algorithms such as using generalized Zubove's method [4] are only applicable for a special class of nonlinear systems.

According to theorem 3, we propose a new method for RDA approximation which is convenient for a large class of nonlinear systems (with time-homogeneous aperiodic chains).

Theorem 3:

Consider nonlinear system (2) with uncertain parameter β, then the support of ϑ_β gives estimated closure of RDA, where ϑ_β is obtained through the following optimization formulation:

$$\vartheta_\beta = \operatorname*{Inf}_{\beta \in B} \vartheta(\beta)$$
$$s.t.\ \vartheta(\beta) = P(\beta)\vartheta(\beta) \tag{12}$$
$$\sum_{i=1}^{N} \vartheta_i(\beta) = 1$$

Where $P(\beta)$ is calculated from Lemma 2 substituting T by T_U.

Proof:

Definition 3 easily implies that

$$\overline{RDA} = \bigcap_{\beta \in B} \overline{DA}(\beta) \tag{13}$$

Where $\overline{DA}(\beta)$ is the closure of domain of attraction of system (2), if we suppose a fixed value for β. According to theorem 2 for a fixed β we have a solution as:

$$\overline{DA}(\beta) = SUPP(\vartheta(\beta)),\ \vartheta(\beta) = P(\beta)\vartheta(\beta),\ \sum_{i=1}^{N} \vartheta_i(\beta) = 1 \tag{14}$$

where $\vartheta(\beta)$ is a vector of invariant measures. In addition from 13 and 14 it is clear that

$$\begin{aligned}\overline{RDA} &= \bigcap_{\beta \in B} SUPP(\vartheta(\beta)) = \bigcap_{\beta \in B} \{A_i \mid \vartheta_i(\beta) \neq 0\} \\ &= \bigcap \{A_i \mid \inf_{\beta \in B} \vartheta_i(\beta) \neq 0\} = SUPP(\inf_{\beta \in B} \vartheta(\beta))\end{aligned} \tag{15}$$

In other words: $\overline{RDA} = SUPP\ \vartheta_\beta$

□

According to theorem 3, we propose an analytic formula to find robust domain of attraction. According to (16), A_i is contained with RDA iff $\vartheta_i(\beta)$ has a nonzero global minimum on B . This global minimum is numerically found using the proposed method of [10]. The advantage of our proposed method is that we express the problem of estimating RDA in the form of a simple optimization problem which is useful for a large class of nonlinear systems.

IV. NUMERICAL EXAMPLES

Consider the following system:

$$\begin{aligned} x_1[k+1] &= (-x_1[k] + (x_2[k])^2 + \beta(-2(x_2[k]) - 2(x_2[k])^2))T_s + x_1[k] \\ x_2[k+1] &= ((3x_1[k] - 2x_2[k] + x_1[k]x_2[k]) \\ &\quad + \beta(-2x_1[k] + 2x_2[k]))T_s + x_2[k] \end{aligned} \tag{16}$$

with uncertain parameter $\beta \in [0, 1]$.

This system is an example of [2] which estimates the RDA of polynomial systems with parameter dependant Lyapunov functions. In figure 1, the green space is the actual RDA in which stability is guaranteed for different values of β and the dashed line is the estimated RDA by [2]. Figure 2 is the estimated RDA obtained from theorem 3. We set $\Omega = [-2,2] \times [-2,2]$, $N_1 = N_2 = 40$, $\Delta_1 = \Delta_2 = .1$, $A_i = [l_{1i}, h_{1i}] \times [l_{2i}, h_{2i}]\ i = 1,..,1600$,

$l_{1i} = -2 + 0.1 remaider\left(\frac{i}{40}\right)$, $l_{2i} = -2 + 0.1 quotient\left(\frac{i}{40}\right)$,

$h_{1i} = l_{1i} + 0.1$ and $h_{2i} = l_{2i} + 0.1$.

Markov matrix in RDA estimation problem is a function of parameter β and it is computed from equation (17). To decrease complexity of cost function $\vartheta(\beta)$ of equation (9) we choose $M_1 = M_2 = 10$ and easily construct $P(\beta)$ matrix:

$$P = [p_{ij}],\ p_{ij} = \frac{\sum_{m_1=0}^{10} \sum_{m_2=0}^{10} [h_T(m_1, m_2, i, j, \beta).h_X(m_1, m_2, i, j)]}{S_D / .16} \tag{17}$$

$where:$

$$h_T = heaviside[(T_1(X^{m_1,m_2}, \beta) - l_{1j}).(h_{1j} - T_1(X^{m_1,m_2}, \beta))]$$
$$heaviside[T_2(X^{m_1,m_2}, \beta) - l_{2j}).(h_{2j} - (T_2(X^{m_1,m_2}, \beta))]$$
$$T_1(X^{m_1,m_2}, \beta) = (x_2^{m_2})^2 - x_1^{m_1} + \beta(-2x_2^{m_2}(1 + x_2^{m_2})) + x_1^{m_1}$$
$$T_2(X^{m_1,m_2}, \beta) = 3x_1^{m_1} - 2x_2^{m_2} + x_1^{m_1}x_2^{m_2} + \beta(-2x_1^{m_1} + 2x_2^{m_2}) + x_2^{m_2}$$
$$h_X(m_1, m_2, i, j) = heaviside\,[(x_1^{m_1} - l_{1i}).(h_{1i} - x_1^{m_1})]\ heaviside\,[(x_2^{m_2} - l_{2i}).(h_{2i} - x_2^{m_2})]$$
$$x_1^{m_k} = 0.4.m_k, x_2^{m_k} = 0.4.m_k$$
$$S_D = 0.01$$

after calculating transient matrix, $\vartheta(\beta)$ is obtained from solving optimization problem (9). To solve this problem we use “fmincon” instruction.

According to the figure 1, for such system both methods define an acceptable estimate of DA but as in Lyapunov based methods estimated DA is usually limited to quadratic structures of Lyapunov function, this method is not applicable for systems with nonquadratic DAs and the result is very conservative. To show it more precisely, we introduce another test system defined in (18). Values of parameters (Ω, N,…) have been stated in the figure description and estimating steps are just like system (17). Figure 3 and 4 show that the Lyapunov based answer is not as appropriate as the previous one. In comparison with Lyapunov based papers, our proposed method does not depend on system structure and if we use a finer partitioning we will obtain a more accurate answer.

The most famous methods of estimating RDA are using common LFs to prove robust local stability [2], Estimating RDA via parameter dependant LFs [2, 3] and RDA estimation through generalized Zubove's method[4]. These methods have some limitations that we overcome theme trough our proposed method. For example in Lyapunov based algorithms RDA of systems with probably time varying uncertainty can be estimated through common LF but finding such a common LF in general is impossible. Parameter dependant LF is applicable only for time invariant uncertainties. In addition, there is not a general LF structure and most literatures use quadratic LF which leads to a conservative estimation of DA as we mentioned in figure 1 to 4. In the other hand to find RDA through generalized Zubove’s method, the viscosity solution of straightforward generalization of classical Zubove's equation is used. This method is concerned with exact determination of DA [5] that causes some limitations. For example, to solve the Zubove's equation, method of characteristic is used but this method requires solution of nonlinear system and in fact the knowledge of DA which is mostly impossible. Another disadvantage of parameter dependant Lyapunove based methods is that the stability is not exactly guaranteed in estimated region. For example in figure 1, one may choose a special β which is not previously considered in intersection, but using it leads to a different estimated RDA. Our proposed method overcomes this limitation by considering all values of parameter β.

In compare with previous estimation algorithms, one disadvantage of Markov modeling is that the estimated RDA (as it can be seen in figure 2) includes real RDA so in boundary partitioning the stability is not guaranteed. To overcome this limitation and have a more accurate estimate of RDA we suggest refining any partition sets which has measure greater than $\frac{1}{N}$, where N is the number of initial partitions.

$$\begin{aligned} x_1[k+1] &= (-x_2[k] + x_1[k]((x_1[k])^2 + (x_2[k])^2 - 1))T_s + x_1[k] \\ x_2[k+1] &= (\beta(x_1[k] + x_2[k]((x_1[k])^2 + (x_2[k])^2 - 1)))T_s + x_2[k] \end{aligned} \tag{18}$$

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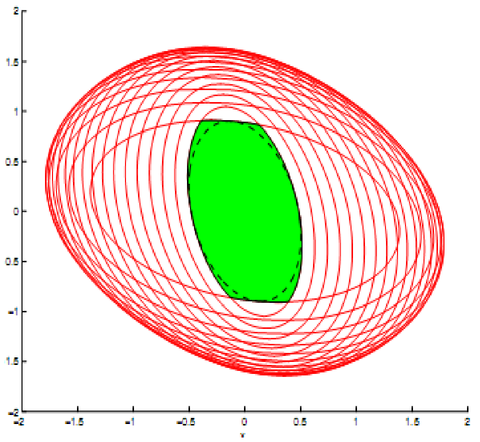

Figure 1. actual RDA of the system for $\beta \in [0,2]$ is shown with green space and estimated region with Lyapunov function is dashed curve

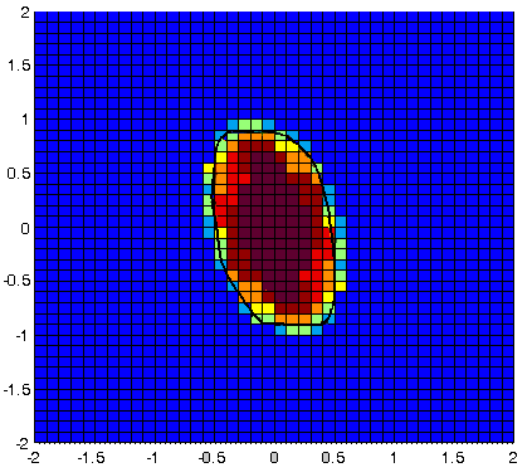

Figure 2. he estimated RDA for $N = 40 \times 40$ partitions in comparison with actual RDA (black curve)

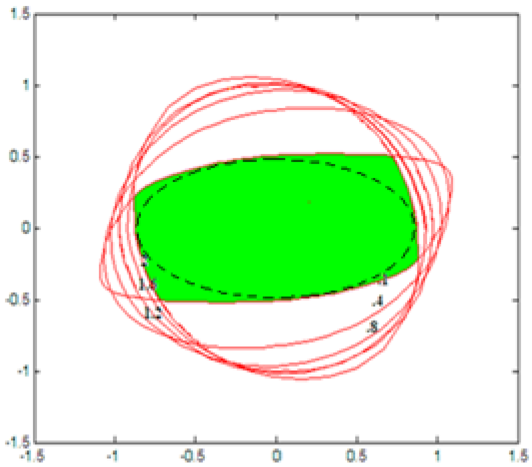

Figure 3. actual RDA of the system for $\beta \in [0.1,2]$ is shown with green space and estimated region with Lyapunov function is dashed curve

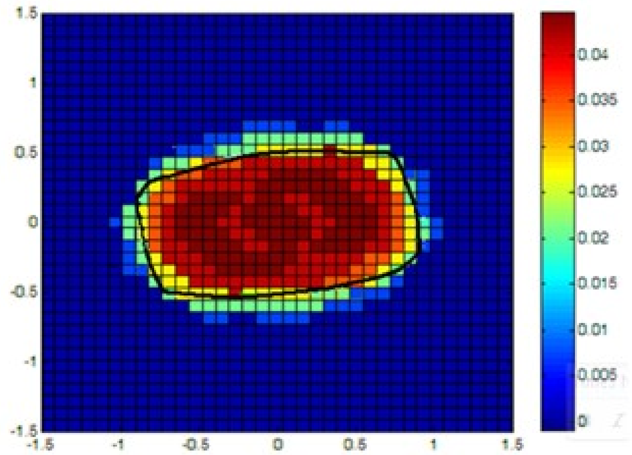

Figure 4. the estimated RDA for $N = 35 \times 35$ partitions in comparison with actual RDA (black curve)

V. CONCLUSIONS

In this work we propose a new method for estimating the RDA based on the average quantities of state space which is obtained from Markov model of system. This model does not use the exact information of real system, in addition it is able to effectively find estimated RDA by solving just a simple optimization problem. Another advantage of this work is its capability of estimating RDA for a large class of nonlinear systems (systems with time-homogeneous aperiodic chains).

One disadvantage of using Markov modeling for RDA estimation is that the estimated RDA includes real RDA so in boundary partitioning the stability is not guaranteed. To overcome this limitation we suggest refining any partition sets which has great invariant measure. Proposing a new algorithm for omitting such boundary partitions will be considered in our future work.

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