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Powers in a class of $\mathcal{A}$-strict standard episturmian words

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1 Introduction

Introduced by Droubay, Justin and Pirillo [8], *episturmian words* are a natural extension of the well-known family of *Sturmian words* (aperiodic infinite words of minimal complexity) to an arbitrary finite alphabet. In this paper, the study of episturmian words is continued in more detail. In particular, for a specific class of episturmian words (a typical element of which we shall denote by $\mathbf{s}$), we will explicitly determine all the integer powers occurring in its constituents. This has recently been done in [6] for Sturmian words, which are exactly the aperiodic episturmian words over a two-letter alphabet.

A finite word w is said to have an integer *power* in an infinite word $\mathbf{x}$ if $w^p = ww\cdots w$ (p times) is a *factor* of $\mathbf{x}$ for some integer $p \geq 2$. Here, our analysis of powers occurring in episturmian words $\mathbf{s}$ hinges on canonical decompositions in terms of their 'building blocks'. Another key tool is a generalization of *singular words*, which were first defined in [17] for the ubiquitous *Fibonacci word*, and later extended to Sturmian words in [15] and the *Tribonacci sequence* in [16]. Our generalized singular words will prove to be useful in the study of factors of episturmian words, just as they have for Sturmian words.

This paper is organized as follows. After some preliminaries (Section 2), we define, in Section 3, a restricted class of episturmian words upon which we will focus for the rest of the paper. A typical element of this class will be denoted by $\mathbf{s}$. In Section 4, we give some simple results which, in turn, lead us to a generalization of *singular words* for episturmian words $\mathbf{s}$. The *index*, i.e., maximal *fractional power*, of the building blocks of $\mathbf{s}$ is then studied in Section 5. Finally, in Section 6, we determine all squares (and subsequently higher powers) occurring in $\mathbf{s}$. The main results are demonstrated via the *k-bonacci word*; a generalization of the Fibonacci word to a k-letter alphabet ($k \geq 2$).

2 Definitions and notations

2.1 Words

Let $\mathcal{A}$ denote a finite alphabet. A (finite) *word* is an element of the *free monoid* $\mathcal{A}^*$ generated by $\mathcal{A}$, in the sense of concatenation. The identity ε of $\mathcal{A}^*$ is called the *empty word*, and the *free semi-group*, denoted by $\mathcal{A}^+$, is defined by $\mathcal{A}^+ := \mathcal{A}^* \setminus \{\varepsilon\}$. Similarly, we define the set $\mathcal{A}^\omega$ of all *infinite words* (or *sequences*) $\mathbf{x} = x_0x_1x_2\cdots$ over $\mathcal{A}$, and set $\mathcal{A}^\infty := \mathcal{A}^* \cup \mathcal{A}^\omega$. If u is a non-empty finite word, then u^ω denotes the *purely periodic* infinite word $uuu\cdots$.

If $w = x_1x_2\ldots x_m \in \mathcal{A}^+$, each $x_i \in \mathcal{A}$, the *length* of w is $|w| = m$ and we denote by $|w|_a$ the number of occurrences of a letter a in w. (Note that $|\varepsilon| = 0$.) The *reversal* of w is $\widetilde{w} = x_mx_{m-1}\ldots x_1$, and if $w = \widetilde{w}$, then w is called a *palindrome*.

A finite word w is a *factor* of $z \in \mathcal{A}^\infty$ if $z = uwv$ for some $u \in \mathcal{A}^*$, $v \in \mathcal{A}^\infty$, and we write $w \prec z$. Further, w is called a *prefix* (resp. *suffix*) of z if $u = \varepsilon$ (resp. $v = \varepsilon$), and we write $w \subseteq_p z$ (resp. $w \subseteq_s z$). An infinite word $\mathbf{x} \in \mathcal{A}^\omega$ is called a *suffix* of $\mathbf{z} \in \mathcal{A}^\omega$ if there exists a word $w \in \mathcal{A}^+$ such that $\mathbf{z} = w\mathbf{x}$. A factor w of a word $z \in \mathcal{A}^\infty$ is *right* (resp. *left*) *special* if wa, wb (resp. aw, bw) are factors of z for some letters $a, b \in \mathcal{A}$, $a \neq b$.

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For $\mathbf{x} \in \mathcal{A}^\omega$, $\Omega(\mathbf{x})$ denotes the set of all its factors, and $\Omega_n(\mathbf{x})$ denotes the set of all factors of $\mathbf{x}$ of length $n \in \mathbb{N}$, i.e., $\Omega_n(\mathbf{x}) := \Omega(\mathbf{x}) \cap \mathcal{A}^n$. Moreover, the *alphabet* of $\mathbf{x}$ is $\mathrm{Alph}(\mathbf{x}) := \Omega(\mathbf{x}) \cap \mathcal{A}$, and we denote by $\mathrm{Ult}(\mathbf{x})$ the set of all letters occurring infinitely often in $\mathbf{x}$. An infinite word $\mathbf{y} \in \mathcal{A}^\omega$ is said to be *equivalent* to $\mathbf{x}$ if $\Omega(\mathbf{y}) = \Omega(\mathbf{x})$, i.e., if $\mathbf{y}$ has the same set of factors as $\mathbf{x}$.

Let $w = x_1x_2\cdots x_m \in \mathcal{A}^*$, each $x_i \in \mathcal{A}$, and let $j \in \mathbb{N}$ with $0 \le j \le m-1$. The *j-th conjugate* of w is the word $C_j(w) := x_{j+1}x_{j+2}\cdots x_m x_1 x_2 \cdots x_j$, and we denote by $\mathcal{C}(w)$ the conjugacy class of w, i.e., $\mathcal{C}(w) := \{C_j(w) \;:\; 0 \le j \le |w|-1\}$. Observe that if w is primitive (i.e., not a power of a shorter word), then w has exactly $|w|$ distinct conjugates.

The *inverse* of $w \in \mathcal{A}^*$, written w^{-1}, is defined by $ww^{-1} = w^{-1}w = \varepsilon$. It must be emphasized that this is merely notation, i.e., for $u, v, w \in \mathcal{A}^*$, the words $u^{-1}w$ and wv^{-1} are defined only if u (resp. v) is a prefix (resp. suffix) of w.

A *morphism on* $\mathcal{A}$ is a map $\psi : \mathcal{A}^* \to \mathcal{A}^*$ such that $\psi(uv) = \psi(u)\psi(v)$ for all $u, v \in \mathcal{A}^*$. It is uniquely determined by its image on the alphabet $\mathcal{A}$.

2.2 Episturmian words

Let $\mathcal{A}$ be an arbitrary finite alphabet. An infinite word $\mathbf{t} \in \mathcal{A}^\omega$ is *episturmian* if $\Omega(\mathbf{t})$ is closed under reversal and $\mathbf{t}$ has at most one right special factor of length n for each $n \in \mathbb{N}$. Moreover, an episturmian word is *standard* if all of its left special factors are prefixes of it.

Let $\mathbf{t}$ be a standard episturmian word over $\mathcal{A}$ and let $u_1 = \varepsilon$, u_2, u_3, $\ldots$ be the sequence of its palindromic prefixes (which exist by results in [8]). Then there exists an infinite word $\Delta(\mathbf{t}) = x_1x_2x_3\ldots$, each $x_i \in \mathcal{A}$, called the *directive word* of $\mathbf{t}$, such that

$$u_{n+1} = (u_n x_n)^{(+)}, \quad n \in \mathbb{N}^+, \tag{2.1}$$

where the *palindromic right-closure* $w^{(+)}$ of a word w is the (unique) shortest palindrome of which w is a prefix (see [7]). The important point here is that a standard episturmian word $\mathbf{t}$ can be constructed as a limit of an infinite sequence of its palindromic prefixes, i.e., $\mathbf{t} = \lim_{n\to\infty} u_n$.

For each letter $a \in \mathcal{A}$, define the morphism Ψ_a on $\mathcal{A}$ by $\Psi_a(a) = a$ and $\Psi_a(x) = ax$ for all $x \in \mathcal{A}\setminus\{a\}$. Further, let us define [12]

$$\mu_n := \Psi_{x_1}\Psi_{x_2}\cdots\Psi_{x_n}, \quad \mu_0 = \mathrm{Id},$$

and

$$h_n := \mu_n(x_{n+1}), \quad n \in \mathbb{N}.$$

Then, we have the following useful formula [12]

$$u_{n+1} = h_{n-1}u_n;$$

and whence, for $n > 1$ and $0 < p < n$,

$$u_n = h_{n-2}h_{n-3}\cdots h_1h_0 = h_{n-2}h_{n-3}\cdots h_{p-1}u_p. \tag{2.2}$$

Lemma 2.1. [12] *For all $n \in \mathbb{N}$,*

(i) *h_n is a primitive word;*

(ii) *$h_n = h_{n-1}$ if and only if $x_{n+1} = x_n$;*

(iii) *if $x_{n+1} \neq x_n$, then u_n is a proper prefix of h_n.* □

Two functions can be defined with regard to positions of letters in a given directive word. For $n \in \mathbb{N}^+$, let $P(n) = \sup\{p < n \;:\; x_p = x_n\}$ if this integer exists, $P(n)$ undefined otherwise. Also, let $S(n) = \inf\{p > n \;:\; x_p = x_n\}$ if this integer exists, $S(n)$ undefined otherwise. By the definitions of palindromic closure and the words u_n, it follows that $u_{n+1} = u_nx_nu_n$ (whence $h_{n-1} = u_nx_n$) if x_n does not occur in u_n, and $u_{n+1} = u_n u_{P(n)}^{-1} u_n$ (whence $h_{n-1} u_{P(n)} = u_n$) if x_n occurs in u_n. Thus, if $P(n)$ exists, then

$$h_{n-1} = h_{n-2} h_{n-3} \cdots h_{P(n)-1}, \quad n \geq 1. \tag{2.3}$$

A standard episturmian word $\mathbf{t}$, or any equivalent (episturmian) word, is said to be $\mathcal{A}$-*strict* (or $|\mathcal{A}|$-*strict*) if $\mathrm{Alph}(\Delta(\mathbf{t})) = \mathrm{Ult}(\Delta(\mathbf{t})) = \mathcal{A}$. The k-strict episturmian words have *complexity* $(k-1)n+1$ for each $n \in \mathbb{N}$ (i.e., $(k-1)n+1$ distinct factors of length n for each $n \in \mathbb{N}$). Such words are exactly the k-letter *Arnoux-Rauzy sequences*, the study of which began in [1].

2.3 Return words

Let $\mathbf{x} \in \mathcal{A}^\omega$ be *recurrent*, i.e., any factor w of $\mathbf{x}$ occurs infinitely often in $\mathbf{x}$. A *return word* [9] of factor w of $\mathbf{x}$ is a factor of $\mathbf{x}$ that begins with w and ends exactly before the next occurrence of w in $\mathbf{x}$. Episturmian words are recurrent and, according to [13, Corollary 4.5], each factor of an $\mathcal{A}$-strict episturmian word has exactly $|\mathcal{A}|$ return words.

3 A class of strict standard episturmian words

Given any infinite sequence $\Delta = x_1 x_2 x_3 \cdots$ over a finite alphabet $\mathcal{A}$, we can define a standard episturmian word having Δ as its directive word (using (2.1)). In this paper, however, we shall only consider a specific family of $\mathcal{A}$-strict standard episturmian words.

Let $\mathcal{A}_k$ denote a k-letter alphabet, say $\mathcal{A}_k = \{a_1, a_2, \ldots, a_k\}$, and suppose $\mathbf{t}$ is a standard episturmian word over $\mathcal{A}_k$. Then the directive word of $\mathbf{t}$ can be expressed as:

$$\Delta(\mathbf{t}) = a_1^{d_1} a_2^{d_2} \cdots a_k^{d_k} a_1^{d_{k+1}} a_2^{d_{k+2}} \cdots a_k^{d_{2k}} a_1^{d_{2k+1}} \cdots ,$$

where the d_i are non-negative integers. In what follows, we restrict our attention to the case when all $d_i > 0$; that is, we shall only study the class of k-strict standard episturmian words $\mathbf{s} \in \mathcal{A}_k^\omega$ with directive words of the form:

$$\Delta = a_1^{d_1} a_2^{d_2} \cdots a_k^{d_k} a_1^{d_{k+1}} a_2^{d_{k+2}} \cdots a_k^{d_{2k}} a_1^{d_{2k+1}} \cdots , \quad d_i > 0. \tag{3.1}$$

This definition of $\mathbf{s}$ will be kept throughout the rest of this paper.

Let us define a sequence $(s_n)_{n \geq 1-k}$ of words associated with $\mathbf{s}$ as follows:

$$\begin{aligned} & s_{1-k} = a_2, \quad s_{2-k} = a_3, \quad \ldots , \quad s_{-1} = a_k, \quad s_0 = a_1, \\ & s_n = s_{n-1}^{d_n} s_{n-2}^{d_{n-1}} \cdots s_0^{d_1} a_{n+1}, \quad 1 \leq n \leq k-1, \\ & s_n = s_{n-1}^{d_n} s_{n-2}^{d_{n-1}} \cdots s_{n-k+1}^{d_{n-k+2}} s_{n-k}, \quad n \geq k. \end{aligned} \tag{3.2}$$

Clearly, s_n is a prefix of s_{n+1} for all $n \geq 0$ (and hence $(|s_n|)_{n \geq 0}$ is a strictly increasing sequence of positive integers).

Example 3.1. It is well-known that the standard Sturmian word c_α of irrational *slope* $\alpha = [0; 1 + d_1, d_2, d_3, \ldots]$, $d_1 \geq 1$, (see [3] for definition) is the standard episturmian word over $\mathcal{A} = \{a, b\}$ with directive word $\Delta(c_\alpha) = a^{d_1} b^{d_2} a^{d_3} b^{d_4} a^{d_5} \cdots$. We have $c_\alpha = \lim_{n \to \infty} s_n$, where $(s_n)_{n \geq -1}$ is the *standard sequence* associated with c_α, defined by

$$s_{-1} = b, \quad s_0 = a, \quad s_n = s_{n-1}^{d_n} s_{n-2}, \quad n \geq 1.$$

This coincides with our definition (3.2) above. Observe that, for all $n \geq 0$, $|s_n| = q_n$, where q_n is the denominator of the n-th convergent to $[0; 1 + d_1, d_2, d_3, \ldots]$.

For all $m \geq 1$, let $L_m := d_1 + d_2 + \cdots + d_m$. Then, writing $\Delta(c_\alpha) = x_1 x_2 x_3 \cdots$ with each $x_i \in \mathcal{A}$, we have $x_{n+1} \neq x_n$ if and only if n is equal to some L_m. One easily deduces that $S(L_m) = L_{m+1} + 1$ and $P(L_{m+1} + 1) = L_m$, and it can also be shown that the h_{L_m} satisfy the same recurrence relation as the q_m. Hence, $|h_{L_m}| = q_m$, and clearly we have $h_{L_m} = s_m$ (see Proposition 3.1, to follow). □

Notation. Hereafter, let $L_n := d_1 + d_2 + \cdots + d_n$ for each $n \geq 1$.

Proposition 3.1. *For any* $n \geq 1$, $s_n = h_{L_n}$. *Moreover,* $\mathbf{s} = \lim_{n\to\infty} s_n$.

Accordingly, the words $(s_n)_{n\geq 1}$ can be viewed as 'building blocks' of $\mathbf{s}$.

Example 3.2. The *Tribonacci sequence* is the standard episturmian word over $\{a, b, c\}$ directed by $(abc)^\omega$. Since all $d_i = 1$, we have $L_n = n$, and hence $h_n = s_n = s_{n-1}s_{n-2}s_{n-3}$, for all $n \geq 1$. □

4 Generalized singular words

Recall the standard Sturmian word c_α of slope $\alpha = [0; 1 + d_1, d_2, d_3, \ldots]$, $d_1 \geq 1$ (Example 3.1). Melançon [15] (also see [4]) introduced the singular words $(w_n)_{n\geq 1}$ of c_α defined by

$$w_n = \begin{cases} as_nb^{-1} & \text{if } n \text{ is odd,} \\ bs_na^{-1} & \text{if } n \text{ is even,} \end{cases}$$

with the convention $w_{-2} = \varepsilon$, $w_{-1} = a$, $w_0 = b$. Let us remark that $s_n = u_{L_n}ab$ (resp. $s_n = u_{L_n}ba$) if n is odd (resp. even).

Singular words are profoundly useful in studying properties of factors of c_α (e.g., [4, 10, 11, 14, 15, 17]). It is for this very reason that we now generalize these words to the case of standard episturmian words $\mathbf{s}$. Firstly, however, we state some basic results concerning the words s_n and u_{L_n}, as detailed in the next section. (Proofs will appear in the extended version of this paper.)

4.1 Useful results

For each $n \geq 0$, set $D_n := u_{L_{n+1}}$. Observe that, for any $m \geq 1$,

$$|D_m| = (d_{m+1} - 1)|s_m| + \sum_{j=0}^{m-1} d_{j+1}|s_j|. \tag{4.1}$$

Indeed, using (2.2), one finds that

$$\begin{aligned} D_m = u_{L_{m+1}} &= h_{L_{m+1}-2}h_{L_{m+1}-3}\cdots h_1h_0 \\ &= h_{L_m}^{d_{m+1}-1}h_{L_{m-1}}^{d_m}h_{L_{m-2}}^{d_{m-1}}\cdots h_{L_1}^{d_2}h_0^{d_1} \\ &= s_m^{d_{m+1}-1}s_{m-1}^{d_m}s_{m-2}^{d_{m-1}}\cdots s_1^{d_2}s_0^{d_1}. \end{aligned} \tag{4.2}$$

Also note that $D_0 = a_1^{d_1-1}$ since $D_0 = u_{d_1} = h_{d_1-2}h_{d_1-3}\cdots h_1h_0 = h_0^{d_1-1}$. For technical reasons, we shall set $D_{-j} := a_{k+1-j}^{-1}$ and $|D_{-j}| = -1$ for $1 \leq j \leq k$.

Proposition 4.1. *Let* $1 \leq i \leq k$. *For all* $n \geq 1 - k$, $a_i \subseteq_s s_n$ *if* $n \equiv i - 1 \pmod{k}$.

Proposition 4.2. *For all* $n \geq 0$, $s_{n+1}D_{n-k+1} = s_nD_n$, *and hence* $|D_n| - |D_{n-k+1}| = |s_{n+1}| - |s_n|$.

Proposition 4.3. *For all* $n \geq 1$, $|s_n| > |D_{n-1}|$.

Recall that the words D_n and s_n are prefixes of $\mathbf{s}$ for all $n \in \mathbb{N}$. Thus, according to Proposition 4.3, the palindromes D_0, D_1, ... , D_{n-1} are prefixes of s_n. In fact, the maximal index i such that D_i is a proper prefix of s_n is $i = n - 1$, which is evident from the following result.

Proposition 4.4. *For all* $n \geq 0$, $D_n = s_n^{d_{n+1}}D_{n-k}$.

Proposition 4.5. *For all* $n \geq 0$, $s_n = D_{n-k}\widetilde{s}_nD_{n-k}^{-1}$.

Remark 4.1. This result shows, in particular, that $\widetilde{s}_n = D_{n-k}^{-1}s_nD_{n-k}$, i.e., $\widetilde{s}_n$ is the $|D_{n-k}|$-th conjugate of s_n for each $n \geq k$. (For $0 \leq n \leq k-1$, $\widetilde{s}_n$ is the $(|s_n| - 1)$-st conjugate of s_n since $\widetilde{s}_n = a_{n+1}s_na_{n+1}^{-1}$.) The following two corollaries are direct results of the above proposition.

Corollary 4.6. *For any $n \geq 0$, the word $\widetilde{s}_n D_{n-k}^{-1}$ is a palindrome. In particular, let $U_n = D_{n-k}$ and $V_n = \widetilde{s}_n D_{n-k}^{-1}$. Then $s_n = U_n V_n$ is the unique factorization of s_n as a product of two palindromes.*

Corollary 4.7. *For all $n \geq 0$, $s_n = D_n \widetilde{s}_n D_n^{-1}$.*

Now, for each $n \in \mathbb{N}$, we define the words $G_{n,r}$ by

$$s_n = D_{n-r} G_{n,r}, \quad 1 \leq r \leq k-1.$$

For example, in the case of Sturmian words c_α, $r = 1$ and $s_n = u_{L_n} G_{n,1}$ for all $n \geq 1$, where $G_{n,1} = ab$ or ba, according to n odd or even, respectively.

Let us note that since $D_{n-r} = a_{k+1+n-r}^{-1}$ for $0 \leq n < r$, we also set

$$G_{n,r} = a_{k+1+n-r} s_n, \quad 0 \leq n < r. \tag{4.3}$$

Proposition 4.8. *For all $n \geq 1$, $s_n s_{n-1} G_{n-1,k-1}^{-1} = s_{n-1} s_n G_{n,1}^{-1}$.*

Remark 4.2. Recall Example 3.1. For c_α with $\alpha = [0; 1+d_1, d_2, d_3 \ldots]$, it is well-known that, for all $n \geq 2$, $s_n s_{n-1}(xy)^{-1} = s_{n-1} s_n (yx)^{-1}$, where $x, y \in \{a, b\}$, $x \neq y$, and $xy \subseteq_s s_{n-1}$. This is known as the *Near-Commutative Property* of the words s_n and s_{n-1}. Because $s_n s_{n-1}(xy)^{-1} = s_n D_{n-2}$ and $s_{n-1} s_n (yx)^{-1} = s_{n-1} D_{n-1}$, Proposition 4.8 is merely an extension of this property to standard episturmian words $\mathbf{s}$. It is also worthwhile noting that Proposition 4.8 shows that s_n is a prefix of $s_{n-1} s_n$.

Proposition 4.2 implies that $|s_{n+1}| - |D_n| = |s_n| - |D_{n-k+1}|$, and hence $|G_{n+1,1}| = |G_{n,k-1}|$. In fact:

Proposition 4.9. *For all $n \geq 1$, $G_{n,1} = \widetilde{G}_{n-1,k-1}$.*

Proposition 4.10. *Let $1 \leq i \leq k$ and $1 \leq r \leq k-1$. For all $n \geq 0$,*

(i) *$a_i \subseteq_p G_{n,r}$ if $n \equiv i + r - 1 \pmod{k}$;*

(ii) *$a_i \subseteq_s G_{n,r}$ if $n \equiv i - 1 \pmod{k}$.*

Hereafter, we set $d_{-j} = 0$ for $j \geq 0$.

4.2 Singular n-words of the r-th kind

By definition of the words $(s_n)_{n \geq 1-k}$ (see (3.2)) and the fact that $\mathbf{s} = \lim_{n \to \infty} s_n$, one deduces that, for any $n \geq 0$, $\mathbf{s}$ can be written as a concatenation of blocks of the form $s_n, s_{n-1}, \ldots, s_{n-k+1}$, i.e.,

$$\begin{aligned} \mathbf{s} = [((& s_n^{d_{n+1}} s_{n-1}^{d_n} \cdots s_{n-k+2}^{d_{n-k+3}} s_{n-k+1})^{d_{n+2}} s_n^{d_{n+1}} \cdots s_{n-k+3}^{d_{n-k+4}} s_{n-k+2})^{d_{n+3}} \\ &(s_n^{d_{n+1}} s_{n-1}^{d_n} \cdots s_{n-k+2}^{d_{n-k+3}} s_{n-k+1})^{d_{n+2}} s_n^{d_{n+1}} \cdots s_{n-k+4}^{d_{n-k+5}} s_{n-k+3}]^{d_{n+4}} \cdots . \end{aligned} \tag{4.4}$$

We shall call this unique decomposition the *n-partition* of $\mathbf{s}$. This will be a useful tool in our subsequent analysis of powers of words occurring in $\mathbf{s}$ (Section 6, to follow).

Remark 4.3. Since each factor of $\mathbf{s}$ has exactly k different return words, two consecutive s_{n+1-i} blocks ($1 \leq i \leq k$) of the n-partition are separated by a word V, of which there are k different possibilities. From now on, it is advisable to keep this observation in mind.

Lemma 4.11. *Let $1 \leq r \leq k-1$. For any $n \in \mathbb{N}^+$, a factor u of length $|s_n|$ of $\mathbf{s}$ is a factor of at least one of the following words:*

- $C_j(s_n)$, $0 \leq j \leq |s_n| - 1$;
- $s_{n-r}^{d_{n-r+1}-1} \cdots s_{n-k+1}^{d_{n-k+2}} s_{n-k} s_{n-1}^{d_n} \cdots s_{n-r+1}^{d_{n-r+2}} s_{n-r} s_n$ *if* $n \geq r$;
- $a_{n+1} s_n a_{n+1}^{-1} a_{n-r+k+1} s_n$ *if* $n < r$.

Remark 4.4. The word $s_{n-r}^{d_{n-r+1}-1}\cdots s_{n-k+1}^{d_{n-k+2}}s_{n-k}s_{n-1}^{d_n}\cdots s_{n-r+1}^{d_{n-r+2}}s_{n-r}$ $(1\le r\le k-1)$ has length $|s_n|$.

Lemma 4.12. *Let $1 \le r \le k-1$. For any $n \ge r$, we have*

$$s_{n-r}^{d_{n-r+1}-1}\cdots s_{n-k+1}^{d_{n-k+2}}s_{n-k}s_{n-1}^{d_n}\cdots s_{n-r+1}^{d_{n-r+2}}s_{n-r} = D_{n-r}\widetilde{G}_{n,r},$$

and for $1 \le n < r$, $a_{n+1}s_na_{n+1}^{-1}a_{n-r+k+1} = \widetilde{G}_{n,r}$.

Whence, it is now plain to see that each word $\widetilde{G}_{n,r}s_n = \widetilde{G}_{n,r}D_{n-r}G_{n,r}$ is a factor of $\mathbf{s}$. We will now partition the set of factors of length $|s_n|$ of $\mathbf{s}$ into k disjoint classes.

Theorem 4.13. *Let $1 \le r \le k-1$. For any $n \in \mathbb{N}^+$, the set of factors of length $|s_n|$ of $\mathbf{s}$ can be partitioned into the following k disjoint classes:*

- $\Omega_n^0 := \mathcal{C}(s_n) = \{C_j(s_n) \ : \ 0 \le j \le |s_n|-1\}$;
- $\Omega_n^r := \{w \in \mathcal{A}_k^* \ : \ |w| = |s_n|$ *and* $w \prec x^{-1}\widetilde{G}_{n,r}D_{n-r}G_{n,r}x^{-1}\}$*, where x is the last letter of $G_{n,r}$.*

That is, $\Omega_{|s_n|}(\mathbf{s}) = \Omega_n^0 \,\dot\cup\, \Omega_n^1 \,\dot\cup \cdots \dot\cup\, \Omega_n^{k-1}$.

Let us remark that $\widetilde{\Omega}_n^r := \{\widetilde{w} \ : \ w \in \Omega_n^r\} = \Omega_n^r$ since $x^{-1}\widetilde{G}_{n,r}D_{n-r}G_{n,r}x^{-1}$ is a palindrome. We shall call the factors of $\mathbf{s}$ in Ω_n^r the *singular n-words of the r-th kind.* Such words will play a key role in our study of powers of words occurring in $\mathbf{s}$.

Evidently, for Sturmian words c_α, $\Omega_n^1 = \{w_n\}$ and we have $\Omega_{|s_n|}(c_\alpha) = \mathcal{C}(s_n) \cup \{w_n\}$.

5 Index

A word of the form $w = (uv)^n u$ is written as $w = z^r$, where $z = uv$ and $r := n + |u|/|z|$. The rational number r is called the *exponent* of z, and w is said to be a *fractional power.*

Now suppose $\mathbf{x}$ is an infinite word. For any $w \prec \mathbf{x}$, the *index* of w in $\mathbf{x}$ is given by the number

$$\mathrm{ind}(w) = \sup\{r \in \mathbb{Q} \ : \ w^r \prec \mathbf{x}\},$$

if such a number exists; otherwise, w is said to have infinite index in $\mathbf{x}$. Furthermore, the greatest number r such that w^r is a prefix of $\mathbf{x}$ is called the *prefix index* of w in $\mathbf{x}$. Obviously, the prefix index is zero if the first letter of w differs from that of $\mathbf{x}$, and it is infinite if and only if $\mathbf{x}$ is purely periodic.

The next two results extend those of Berstel [2].

Lemma 5.1. *For all $n \ge 1$, the prefix index of s_n in $\mathbf{s}$ is $1 + d_{n+1} + |D_{n-k}|/|s_n|$.*

Lemma 5.2. *For all $n \ge 1$, the index of s_n as a factor of $\mathbf{s}$ is $\mathrm{ind}(s_n) = 2 + d_{n+1} + |D_{n-k}|/|s_n|$, and hence $\mathbf{s}$ contains cubes.*

6 Powers occurring in s

For each $m, l \in \mathbb{N}$ with $l \ge 2$, let us define the following set of words:

$$\mathcal{P}(m; l) := \{w \in \mathcal{A}_k^* \ : \ |w| = m, \ w^l \prec \mathbf{s}\},$$

where $\mathbf{s}$ is the k-strict standard episturmian word over $\mathcal{A}_k = \{a_1, a_2, \ldots, a_k\}$ with directive word Δ given by (3.1). Also, let $p(m; l) := |\mathcal{P}(m; l)|$.

The next theorem is a generalization of Theorem 1 in [6]. It gives all the lengths m such that there is a non-trivial power of a word of length m in $\mathbf{s}$. Firstly, let us define the following k sets of lengths for fixed $n \in \mathbb{N}^+$:

$$\mathcal{D}_1(n) := \{r|s_n| \ : \ 1 \le r \le d_{n+1}\},$$
$$\mathcal{D}_i(n) := \{|s_n^r s_{n-1}^{d_n}\cdots s_{n+2-i}^{d_{n+3-i}}s_{n+1-i}| \ : \ 1 \le r \le d_{n+1}\}, \quad 2 \le i \le k-1,$$
$$\mathcal{D}_k(n) := \{|s_n^r s_{n-1}^{d_n}\cdots s_{n+2-k}^{d_{n+3-k}}s_{n+1-k}| \ : \ 1 \le r \le d_{n+1}-1\}.$$

Theorem 6.1. *Let m, $n \in \mathbb{N}^+$ be such that $|s_n| \leq m < |s_{n+1}|$ and suppose $m \notin \bigcup_{i=1}^{k} \mathcal{D}_i(n)$. Then $p(m;l) = 0$ for all $l \geq 2$.*

Remark 6.1. Put simply, the above theorem states that if $m \notin \bigcup_{i=1}^{k} \mathcal{D}_i(n)$ for some n, then there are no l-th powers of words of length m in $\mathbf{s}$ for any $l \geq 2$. Equivalently, if $w^l \prec \mathbf{s}$ with $|s_n| \leq |w| < |s_{n+1}|$, then $|w| \in \bigcup_{i=1}^{k} \mathcal{D}_i(n)$. For instance, if $k = 3$ and $|s_n| \leq m < |s_{n+1}|$ with

$$m \notin \{|s_n^r|, |s_n^r s_{n-1}| : 1 \leq r \leq d_{n+1}\} \cup \{|s_n^r s_{n-1}^{d_n} s_{n-2}| : 1 \leq r \leq d_{n+1} - 1\},$$

then $p(m;l) = 0$ for all $l \geq 2$. For the particular case of the Tribonacci sequence, this implies that if w^l is a factor, then $|w| \in \{|s_n|, |s_n| + |s_{n-1}|\}$ for some n, where the lengths $(|s_i|)_{i \geq 0}$ are the *Tribonacci numbers*: $T_0 = 1$, $T_1 = 2$, $T_2 = 4$, $T_i = T_{i-1} + T_{i-2} + T_{i-3}$, $i \geq 3$.

The proof of Theorem 6.1 requires several lemmas (Lemmas 6.2–6.4 below). Let us first observe that in the n-partition of $\mathbf{s}$ (see (4.4)) to the left of each s_n block, there is an s_{n+1-j} block for some $j \in [1,k]$. Also note that each s_{n+1-j} is a prefix of s_n. Furthermore, to the left of each s_{n+1-i} block is another s_{n+1-i} block or an s_{n+2-i} block, for each $i \in [2,k]$.

Lemma 6.2. *Let $n \in \mathbb{N}^+$. Consider a word $w \prec \mathbf{s}$ of the form $w = us_nv$ for some words u, $v \in \mathcal{A}_k^*$, $u \neq \varepsilon$.*

(i) *If $w = u_1u_2$, where $u_1 \subseteq_s s_{n+1-i}$ for some $i \in [1,k]$ and $u_2 \subseteq_p s_n$, then $u_1 = u$.*

(ii) *If $w = u_1 s_{n+1-i} u_2$ for some $i \in [2,k]$, where $u_1 \subseteq_s s_{n+2-i}$ and $u_2 \subseteq_p s_n$, then $u_1 = u$ or $u_1 s_{n+1-i} = u$.*

(iii) *If $w = u_1 s_{n+1-i} u_2$ for some $i \in [2,k-1]$, where $u_1 \subseteq_s s_{n+1-i}$ and $u_2 \subseteq_p s_n$, then $u_1 = u$ or $u_1 s_{n+1-i} = u$.*

Lemma 6.3. *Let $c \in \mathcal{A}_k$ and $n \in \mathbb{N}$ be fixed. Consider an occurrence of cs_n in $\mathbf{s}$. Then the letter c is the last letter of a block s_{n+1-i} of the n-partition of $\mathbf{s}$, for some $i \in [1,k]$, and the integer i (equiv. the block s_{n+1-i}) is uniquely determined by c. In particular, in every occurrence of $s_{n+1-i}s_n$ in $\mathbf{s}$, the word s_{n+1-i} is a block in the n-partition of $\mathbf{s}$.*

That is, occurrences of words w containing cs_n ($c \in \mathcal{A}_k$) must be aligned to the n-partition of $\mathbf{s}$. Now we have an analogue of Lemma 3.5 in [5]:

Lemma 6.4. *Let $n \in \mathbb{N}^+$ and suppose $u \prec \mathbf{s}$ with $|s_n| \leq |u| < |s_{n+1}|$. Then the following assertions hold.*

(1) *For all $i \in [1,k]$, there is at most one position in s_{n+1-i} such that any occurrence of u in $\mathbf{s}$ which starts in some s_{n+1-i} block of the n-partition of $\mathbf{s}$ must start at this particular position in s_{n+1-i}.*

(2) *For all $i \in [1,k-1]$, if u can start at position l in s_{n+1-i} and at position m in s_{n-i}, then $l = m$.*

Notation. Given $l \in \mathbb{N}$ and $w \in \mathcal{A}_k^*$, denote by $\mathrm{Pref}_l(w)$ the prefix of w of length l if $|w| \geq l$, w otherwise. Likewise, denote by $\mathrm{Suff}_l(w)$ the suffix of w of length l if $|w| \geq l$, w otherwise. Recall that Ω_n^r denotes the set of singular n-words of the r-th kind ($1 \leq r \leq k-1$), as defined in Theorem 4.13.

Lemma 6.5. *Let $n \in \mathbb{N}^+$ and suppose $w \in \Omega_{n+1-i}^1$ for some $i \in [1,k-1]$. Then w begins with $v := \mathrm{Suff}_l(x^{-1}\widetilde{G}_{n+1-i,1})$ for some $l \in \mathbb{N}$ with $1 \leq l \leq |G_{n+1-i,1}| - 1$. Moreover, the word vs_{n+1-i} occurs at position p in $\mathbf{s}$ if and only if the n-partition of $\mathbf{s}$ contains an s_n starting at postion $p+l$ and an s_{n-i} ending at position $p+l-1$. In particular, w occurs at exactly those positions where vs_{n+1-i} occurs in $\mathbf{s}$.*

Note. It is assumed that $n \geq i$.

Consider two distinct occurrences of a factor w in $\mathbf{s}$, say

$$\mathbf{s} = uw\mathbf{v} = u'w\mathbf{v}', \quad |u'| > |u|,$$

where $\mathbf{v}$, $\mathbf{v}' \in \mathcal{A}_k^\omega$. These two occurrences of w in $\mathbf{s}$ are said to be *positively separated* (or *disjoint*) if $|u'| > |uw|$, in which case $u' = uwz$ for some $z \in \mathcal{A}_k^+$, and hence $\mathbf{s} = uwzw\mathbf{v}'$.

Lemma 6.6. *For any $n \in \mathbb{N}^+$, successive occurrences of a singular word $w \in \bigcup_{j=1}^{k-1} \Omega_n^j$ in $\mathbf{s}$ are positively separated.*

The next lemma follows from Lemmas 5.2, 6.5 and 6.6.

Lemma 6.7. *Let $n \in \mathbb{N}^+$ and suppose $u \prec \mathbf{s}$ with $|u| = |s_n|$. Then $u^2 \prec \mathbf{s}$ if and only if $u \in \mathcal{C}(s_n)$. In particular, if u is a singular word of any kind of $\mathbf{s}$, then $u^2 \not\prec \mathbf{s}$.*

More generally, we have the following result.

Lemma 6.8. *Let $n \in \mathbb{N}^+$ and suppose $u^2 \prec \mathbf{s}$ with $|s_n| \leq |u| < |s_{n+1}|$. Then u does not contain a singular word from the set Ω_{n+1-i}^1 for any $i \in [1, k-1]$.*

6.1 Squares

The next two main theorems concern squares of factors of $\mathbf{s}$ of length $m < d_1 + 1 = |s_1|$ and length $m \geq |s_1|$, respectively.

A letter a in a finite or infinite word w is *separating for w* if any factor of length 2 of w contains the letter a. For example, a is separating for the infinite word $(aaba)^\omega$. If a is separating for an infinite word $\mathbf{x}$, then it is clearly separating for any factor of $\mathbf{x}$. According to [8, Lemma 4], since the standard episturmian word $\mathbf{s}$ begins with a_1, the letter a_1 is separating for $\mathbf{s}$ and its factors. Moreover, a_1 occurs in runs of length d_1 or $d_1 + 1$ in $\mathbf{s}$ (inspect the 0-partition of $\mathbf{s}$), and the following is deduced:

Theorem 6.9. *For $1 \leq r \leq d_1$, we have*

$$p(r; 2) = \begin{cases} 1 & \text{if } r \leq (d_1+1)/2, \\ 0 & \text{if } r > (d_1+1)/2. \end{cases}$$

In particular, $\mathcal{P}(r; 2) = \{(a_1^r)^2\}$ for $r \leq (d_1+1)/2$, and $\mathcal{P}(r; 2) = \emptyset$ for $r > (d_1+1)/2$.

Theorem 6.10. *Let n, $r \in \mathbb{N}^+$.*

(i) *For $1 \leq r \leq d_{n+1}$,*

$$p(|s_n^r|; 2) = \begin{cases} |s_n| & \text{if } 1 \leq r < 1 + d_{n+1}/2, \\ |D_{n-k}| + 1 & \text{if } d_{n+1} \text{ is even and } r = 1 + d_{n+1}/2, \\ 0 & \text{if } 1 + d_{n+1}/2 < r \leq d_{n+1}. \end{cases} \tag{6.1}$$

That is,

$$\mathcal{P}(|s_n^r|; 2) = \begin{cases} \{C_j(s_n^r) \ : \ 0 \leq j \leq |s_n| - 1\} & \text{if } 1 \leq r < 1 + d_{n+1}/2, \\ \{C_j(s_n^r) \ : \ 0 \leq j \leq |D_{n-k}|\} & \text{if } d_{n+1} \text{ is even and } r = 1 + d_{n+1}/2, \\ \emptyset & \text{if } 1 + d_{n+1}/2 < r \leq d_{n+1}. \end{cases} \tag{6.2}$$

(ii) *For $1 \leq r \leq d_{n+1}$ and $i \in [2, k]$ (with $r \neq d_{n+1}$ if $i = k$), we have*

$$p(|s_n^r s_{n-1}^{d_n} \cdots s_{n+2-i}^{d_{n+3-i}} s_{n+1-i}|; 2) = |D_{n+1-i}| + 1. \tag{6.3}$$

That is,

$$\mathcal{P}(|s_n^r s_{n-1}^{d_n} \cdots s_{n+2-i}^{d_{n+3-i}} s_{n+1-i}|; 2) = \{C_j(s_n^r s_{n-1}^{d_n} \cdots s_{n+2-i}^{d_{n+3-i}} s_{n+1-i}) \ : \ 0 \leq j \leq |D_{n+1-i}|\}. \tag{6.4}$$

Remark 6.2. For standard Sturmian words c_α, we have $s_n = D_{n-1}xy$, where x, $y \in \{a, b\}$ $(x \neq y)$, and hence $|D_{n-1}| = q_n - 2$ for all $n \geq 1$. Accordingly, Theorem 6.10 agrees with Theorem 3 in [6] for the case of a two-letter alphabet.

Theorem 6.9 is trivial, whereas the proof of Theorem 6.10 requires the following two lemmas.

Lemma 6.11. *Let $n \in \mathbb{N}^+$ and let $u^2 = u^{(1)}u^{(2)}$ be an occurrence of u^2 in $\mathbf{s}$, where $|s_n| \le |u| < |s_{n+1}|$.*

(i) *For all $n \ge 1$, if $|u| = |s_n^r|$ with $1 \le r \le d_{n+1}$, then $u^{(1)}$ begins in an s_n block of the n-partition of $\mathbf{s}$. Moreover, u^2 is a factor of $s_n^{d_{n+1}+2}s_n v^{-1} = s_n^{d_{n+1}+2}D_{n-k}$, where $|v| = |s_n| - |D_{n-k}|$.*

(ii) *Let $i \in [2, k-1]$. For all $n \ge i-1$, if $|u| = |s_n^r s_{n-1}^{d_n} \cdots s_{n+2-i}^{d_{n+3-i}} s_{n+1-i}|$ with $1 \le r \le d_{n+1}$, then $u^{(1)}$ starts in an s_n block and contains an s_{n+1-i} block that is followed by an s_n block in the n-partition of $\mathbf{s}$. Moreover, u^2 is a factor of $(s_n^r s_{n-1}^{d_n} \cdots s_{n+2-i}^{d_{n+3-i}} s_{n+1-i})^2 D_{n+1-i}$, which is a factor of*

$$(s_n^{d_{n+1}} s_{n-1}^{d_n} \cdots s_{n+2-i}^{d_{n+3-i}} s_{n+1-i})^2 D_{n+1-i}.$$

(iii) *For all $n \ge k-1$, if $|u| = |s_n^r s_{n-1}^{d_n} \cdots s_{n+2-k}^{d_{n+3-k}} s_{n+1-k}|$ with $1 \le r \le d_{n+1}-1$, then $u^{(1)}$ starts in an s_n block and contains an s_{n+1-k} block of the n-partition of $\mathbf{s}$. Moreover, u^2 is a factor of $(s_n^r s_{n-1}^{d_n} \cdots s_{n+2-k}^{d_{n+3-k}} s_{n+1-k})^2 D_{n+1-k}$, which is a factor of*

$$s_{n+1}^2 = (s_n^{d_{n+1}} s_{n-1}^{d_n} \cdots s_{n+2-k}^{d_{n+3-k}} s_{n+1-k})^2.$$

Lemma 6.12. *For all n, $r \in \mathbb{N}^+$ and $i \in [2, k]$, the word $v := s_n^r s_{n-1}^{d_n} \cdots s_{n+2-i}^{d_{n+3-i}} s_{n+1-i}$ is primitive.*

6.2 Cubes and higher powers

Our subsequent analysis of cubes and higher powers occurring in $\mathbf{s}$ is now an easy task due to the above consideration of squares. Extending Theorem 6.10 (see Theorem 6.14 below), only requires the following lemma, together with arguments used in the proof of Theorem 6.10.

Lemma 6.13. *Let $n \in \mathbb{N}^+$ and suppose $u^3 \prec \mathbf{s}$ with $|s_n| \le |u| < |s_{n+1}|$. Then u^3 does not contain a singular word from the set Ω_{n+1-i}^1 for any $i \in [1, k-1]$.*

Theorem 6.14. *Let n, r, $l \in \mathbb{N}^+$, $l \ge 3$.*

(i) *For $1 \le r \le d_{n+1}$,*

$$p(|s_n^r|; l) = \begin{cases} |s_n| & \text{if } 1 \le r < (d_{n+1}+2)/l, \\ |D_{n-k}| + 1 & \text{if } r = (d_{n+1}+2)/l, \\ 0 & \text{if } (d_{n+1}+2)/l < r \le d_{n+1}. \end{cases} \tag{6.5}$$

That is,

$$\mathcal{P}(|s_n^r|; l) = \begin{cases} \{C_j(s_n^r) \ : \ 0 \le j \le |s_n| - 1\} & \text{if } 1 \le r < (d_{n+1}+2)/l, \\ \{C_j(s_n^r) \ : \ 0 \le j \le |D_{n-k}|\} & \text{if } r = (d_{n+1}+2)/l, \\ \emptyset & \text{if } (d_{n+1}+2)/l < r \le d_{n+1}. \end{cases} \tag{6.6}$$

(ii) *For $1 \le r \le d_{n+1}$ and $i \in [2, k]$ (with $r \ne d_{n+1}$ if $i = k$), we have*

$$p(|s_n^r s_{n-1}^{d_n} \cdots s_{n+2-i}^{d_{n+3-i}} s_{n+1-i}|; l) = 0. \tag{6.7}$$

Example 6.1. Let us define the *k-bonacci word* to be the standard episturmian word $\boldsymbol{\eta}_k \in \mathcal{A}_k^\omega$ with directive word $(a_1a_2\cdots a_k)^\omega$. Since all $d_i = 1$, we have $s_n = s_{n-1}s_{n-2}\cdots s_{n-k}$ for all $n \ge 1$ (and the lengths $|s_n|$ are the k-bonacci numbers). Thus, for fixed $n \in \mathbb{N}^+$ and $l \ge 2$, if $w^l \prec \boldsymbol{\eta}_k$ with $|s_n| \le |w| < |s_{n+1}|$, then we necessarily have $|w| = |s_n| + |s_{n-1}| + \cdots + |s_{n+1-i}|$ for some $i \in [1, k-1]$ (by Theorem 6.1). The preceding theorems reveal that

$$\mathcal{P}(1; 2) = \{a_1\}, \quad \mathcal{P}(|s_n|; 2) = \mathcal{C}(s_n) = \Omega_n^0 \quad \text{and} \quad \mathcal{P}(|s_n|; 3) = \{C_j(s_n) \ : \ 0 \le j \le |D_{n-k}|\}.$$

Furthermore, for each $i \in [2, k-1]$, we have

$$\mathcal{P}(|s_ns_{n-1}\cdots s_{n+1-i}|; 2) = \{C_j(s_ns_{n-1}\cdots s_{n+1-i}) \ : \ 0 \le j \le |D_{n+1-i}|\}.$$

All other $\mathcal{P}(|w|; l) = \emptyset$, $l \ge 2$. In particular, k-bonacci words are 4-power free. □

7 Concluding remarks

Theorems 6.9, 6.10 and 6.14 also suffice to describe all integer powers occurring in any (episturmian) word $\mathbf{t} \in \mathcal{A}_k^\omega$ that is equivalent to $\mathbf{s}$. (See [12, Theorem 3.10] for a definition of such $\mathbf{t}$.) The problem of determining all integer powers occurring in general standard episturmian words (with not all d_i necessarily positive) remains open.

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