Directive words of episturmian words

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Introduction

- G. Richomme mentionned in the previous talk works about quasiperiodicity of episturmian words [GLR2008].
- He presented morphic decompositions of episturmian words which led us to the result.
- Doing this study, we wanted to know more about morphic decompositions of episturmian words: unicity, equivalences, . . .

Plan

1. Directive words of episturmian words
   - Episturmian words and morphisms
   - Morphic decompositions of episturmian words
   - Spinned words

2. Many questions about directive words of episturmian words
   - Do all spinned infinite word direct a unique episturmian word?
   - Characterization of the words directing a common episturmian word
   - When does an episturmian word have a unique directive word?

Directive words of episturmian words
Episturmian words and morphisms

- Notions seen in the previous talk:

  Let the morphisms L_a and R_a where, for all $a \in A$

  $$L_a : \begin{cases} a \mapsto a \\ b \mapsto ab \end{cases} \quad R_a : \begin{cases} a \mapsto a \\ b \mapsto ba \end{cases} \quad \text{for all } b \neq a \in A.$$

- Notation:

  Morphisms obtained by composition of elements of $\mathcal{L}_A = \{L_a \mid a \in A\}$ and $\mathcal{R}_A = \{R_a \mid a \in A\}$ are called "pure episturmian morphisms".

[Justin Pirillo 2002]

- A word is episturmian if and only if it can be infinitely decomposed over pure episturmian morphisms.
- A word is epistandard if and only if it can be infinitely decomposed over morphisms in $\mathcal{L}_A$ (pure epistandard morphisms).

Directive words of episturmian words

Morphic decompositions of episturmian words

[Justin Pirillo 2002]

A word is epistandard if and only if it can be infinitely decomposed over pure epistandard morphisms.

Example

An example of morphic decomposition of a Sturmian word: Fibonacci

$$w = abaababaabaababaababaabaababaaba\ldots \qquad L_a \begin{cases} a \mapsto a \\ b \mapsto ab \end{cases}$$

$$w = L_a(b$$

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$w = L_a(babb$

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$$w = L_aL_b(abaababaabaa\dots)$$

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$$L_a \begin{cases} a \mapsto a \\ b \mapsto ab \end{cases} \qquad L_b \begin{cases} a \mapsto ba \\ b \mapsto b \end{cases}$$

$$w = abaababaabaababaababaabaababaaba\ldots$$

$$w = L_a(babbababbabbababbaba\ldots)$$

$$w = L_aL_b(abaababaabaa\ldots)$$

$$w = L_aL_bL_a(babbaba\ldots)$$

$$w = L_aL_bL_aL_b(abaa\ldots)$$

- $(L_aL_b)^\omega$ is the *morphic decomposition* of w.
- $\Delta = (ab)^\omega$ is the *directive word* of F.

Directive words of episturmian words
Spinned words

- To translate a morphic decomposition in terms of *spinned word* :

$$L_x \rightarrow x \qquad (x \text{ with spin } L)$$
$$R_x \rightarrow \bar{x} \qquad (x \text{ with spin } R)$$

Example

- $(R_a R_b L_a)^\omega$ *morphic decomposition* of a word $w \Rightarrow \breve{\Delta} = (\bar{a}\bar{b}a)^\omega$ *directive word* of w.

- Conversely

$$x \rightarrow \mu_x = L_x$$
$$\bar{x} \rightarrow \mu_{\bar{x}} = R_x$$
$$\mu_w = \mu_{w_1} \ldots \mu_{w_n} \ldots \text{ for } w = w_1 \ldots w_n \ldots$$

- A word over $A \cup \bar{A}$ is called a *spinned word*.
- The opposite $\bar{w}$ of a spinned word w is obtained by exchanging all spins in w.

Example

- $\overline{\bar{a}\bar{b}a\bar{c}} = ab\bar{a}c$
- $\overline{(a^\omega)} = \bar{a}^\omega$

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Many questions

1 Do all spinned infinite words direct a unique episturmian word?

2 Is it possible that two different spinned words direct a common episturmian word? What are their forms in this case?

3 In general an episturmian word has several equivalent directive words. When does an episturmian word have a unique directive word?

Many answers...

1. Do all spinned infinite word direct a unique episturmian word?
   - Justin Pirillo 2002

Do all spinned infinite words direct a unique episturmian word?

NO !

[Justin-Pirillo 2002]

1. Any ultimately R-spinned word $\breve{\Delta}$ directs exactly one episturmian word for each letter in $Ult(\breve{\Delta})$.

   Example

   $\Delta = \bar{a}(\bar{b}\bar{c}\bar{a})^{\omega}$ directs an episturmian word starting with a.
   But $\Delta = \bar{a}\bar{b}(\bar{c}\bar{a}\bar{b})^{\omega}$ also directs an episturmian word starting with b.

2. Any spinned infinite word $\breve{\Delta}$ having infinitely many L-spinned letters directs a unique episturmian word beginning with the left-most letter having spin L in $\breve{\Delta}$.
   - $\Rightarrow$ Any L-spinned infinite word directs a unique epistandard word.

Many answers... but

2 Is it possible that two different spinned words direct a common episturmian word? What are their forms in this case?

- Almost all answers by Justin and Pirillo (2004). They provide:
  - all answers in the aperiodic cases
  - partial answers in the periodic cases

  But...
  - Their results are disseminated into 6 different results
  - their results allow to verify if two spinned words are directive-equivalent, but they don't provide an easy way to check it just by "seeing" the spinned words.
- Our work:
  - We complete the characterization of spinned words directing a common episturmian word.
  - We unify the results so that there is less distinct cases
  - We give a more "easy to check" way to verify if two spinned words direct a common episturmian word, by providing explicitely the possible forms of the spinned words.

Block-equivalence of finite words

- Two finite spinned words are *block-equivalent* ($\equiv$) if we can pass from one to the other by a chain of block-transformations, that is:

$$x v \bar{x} \text{ is replaced by } \bar{x} \bar{v} x \text{ for } x \in A,\ v \in A^* \setminus \{x\}.$$

Example

$$\bar{b}\bar{a}b\bar{c}b\bar{a}\bar{c} \longrightarrow ba\bar{b}\bar{c}b\bar{a}\bar{c} \longrightarrow babc\bar{b}\bar{a}\bar{c},$$
$$babc\bar{b}\bar{a}\bar{c} \longrightarrow ba\bar{b}\bar{c}b\bar{a}\bar{c} \longrightarrow \bar{b}\bar{a}b\bar{c}b\bar{a}\bar{c}.$$

Block-equivalence vs presentation of the episturmian monoid

Theorem [R2003, see also JP2004]

The monoid of *pure episturmian morphisms* with $\{L_\alpha, R_\alpha \mid \alpha \in A\}$ as set of generators has the following presentation:

$$R_{a_1} R_{a_2} \dots R_{a_k} L_{a_1} = L_{a_1} L_{a_2} \dots L_{a_k} R_{a_1}$$

where $k \geq 1$ integer and $a_1, \dots, a_k \in A$ with $a_1 \neq a_i$ for all i, $2 \leq i \leq k$.

Example

$$\begin{aligned}
& R_a R_b L_c R_b L_b R_a R_c R_b R_a R_c L_a \\
= \quad & R_a R_b L_c R_b L_b R_a R_c R_b L_a L_c R_a \\
= \quad & R_a R_b L_c L_b R_b R_a R_c R_b L_a L_c R_a \\
= \quad & R_a R_b L_c L_b R_b R_a R_c R_b R_a R_c L_a \\
= \quad & R_a R_b L_c L_b R_b L_a L_c L_b L_a L_c R_a \\
\dots &
\end{aligned}$$

Block-equivalence of infinite words

- Justin and Pirillo use of a relation "leads to" ($\rightsquigarrow$).
- $\Delta_1 \rightsquigarrow \Delta_2$ if there exist infinitely many prefixes f_i of Δ_1 and g_i of Δ_2 with the g_i of strictly increasing lengths, and such that, for all i, $|g_i| \leq |f_i|$ and $f_i \equiv g_i c_i$ for a suitable spinned word c_i.
- $\Delta_1 \equiv \Delta_2$ if $\Delta_1 \rightsquigarrow \Delta_2$ and $\Delta_2 \rightsquigarrow \Delta_1$.
- This approach doesn't allow to "see" straight away if two spinned words direct a common word.

The periodic case

Let $\breve{\Delta}$ be a spinned version of an L-spinned word Δ, t be an episturmian word directed by $\breve{\Delta}$ and s the epistandard word directed by Δ. Then t and s have the same set of factors.

Moreover **t** periodic $\Leftrightarrow$ **s** periodic $\Leftrightarrow$ $|Ult(\Delta)| = 1$ [Justin Pirillo 2002].

Justin Pirillo 2004

$\breve{\Delta}_1 = \breve{w}\breve{y}a^\omega$
$\breve{\Delta}_2 = \hat{w}\hat{y}\bar{a}^\omega$
$\breve{\Delta}_1$ and $\breve{\Delta}_2$ are directive-equivalent iff there exist sequences of letters $(\breve{a}_n)_{n\geq 1}$ and $(\hat{a}_n)_{n\geq 1}$ such that $\breve{w}\breve{y}\prod_{n\geq 1}\breve{a}_n \equiv \hat{w}\hat{y}\prod_{n\geq 1}\hat{a}_n$.

The periodic case

It is not the only periodic case!

[GLR2008]

$\Delta_1 = w\mathbf{x}$ and $\Delta_2 = w'\mathbf{y}$ where w, w' are spinned words, x and y are letters, and $\mathbf{x} \in \{x, \bar{x}\}^\omega$, $\mathbf{y} \in \{y, \bar{y}\}^\omega$ are spinned infinite words such that $\mu_w(x) = \mu_{w'}(y)$. Then Δ_1 and Δ_2 are directive-equivalent.

Example

$(ab)^\omega = L_a(b^\omega) = R_b(a^\omega)$ is directed by

- ab^ω
- $\bar{b}a^\omega$
- $a\bar{b}^\omega$
- $ab\bar{b}^\omega$
- $ab\bar{b}bb\bar{b}$
- words in $a\{b, \bar{b}\}^\omega$

The aperiodic cases

Case where one of the words is L-spinned

Justin Pirillo 2004

Let Δ be an L-spinned infinite word. Then Δ and $\bar{\Delta}$ do not direct a common right-infinite episturmian word.

The aperiodic cases

Case of *wavy* spinned words (with infinitely many letters of spin L and R)

[JP2004] If an aperiodic episturmian word is directed by two spinned words Δ_1 and Δ_2, then Δ_1 and Δ_2 are spinned versions of a common L-spinned word.

JP2004

When
Δ_1 and Δ_2 do not have any common prefix modulo $\equiv$,
$\exists x$ such that $\Delta_1 = xw$ and $\Delta_2 = \bar{x}w'$

Then $\quad \Delta_1 \equiv \Delta_2 \quad \Rightarrow \quad \Delta_1 = x \prod_{n\geq 1}(v_n \check{x}_n), \qquad \Delta_2 = \bar{x} \prod_{n\geq 1}(\bar{v}_n \hat{x}_n),$

where $(v_n)_{n\geq 1}$ are x-free L-spinned words,
and $(\check{x}_n)_{n\geq 1}$, $(\hat{x}_n)_{n\geq 1}$ in $\{x, \bar{x}\}$ such that $(\check{x}_n)_{n\geq 1}$ contains infinitely many $\bar{x}$,
and $(\hat{x}_n)_{n\geq 1}$ contains infinitely many x.

The aperiodic cases

Case where each word is ultimately L-spinned or R-spinned

Justin Pirillo 2004

Let Δ_1 and Δ_2 be spinned versions of a common word $\Delta \in \mathcal{A}^\omega$. If there exist spinned words w_1, w_2 and an L-spinned infinite word Δ' such that

$$\Delta_1 = w_1\Delta' \text{ and } \Delta_2 = w_2\Delta' \text{ (resp. } \Delta_1 = w_1\bar{\Delta}' \text{ and } \Delta_2 = w_2\bar{\Delta}'\text{)},$$

then

$$\Delta_1,\ \Delta_2 \text{ are directive-equivalent if and only if } \mu_{w_1} = \mu_{w_2}.$$

Our characterization of directive-equivalent words

[GLR2008]

Given two spinned infinite words Δ_1 and Δ_2, the following assertions are equivalent.

i) Δ_1 and Δ_2 direct a common right-infinite episturmian word;

ii) One of the following cases holds for some i, j such that $\{i, j\} = \{1, 2\}$:

1. $\Delta_i = \prod_{n \geq 1} v_n$, $\Delta_j = \prod_{n \geq 1} z_n$ where $\mu_{v_n} = \mu_{z_n}$ for all $n \geq 1$;

Example

$abc\bar{a}.\bar{a}\bar{b}\bar{a}a.\bar{b}\bar{c}b\ldots$

$\bar{a}\bar{b}\bar{c}a.aba\bar{a}.bc\bar{b}\ldots$

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2. $\Delta_i = wx \prod_{n \geq 1}(v_n \breve{x}_n)$, $\Delta_j = w' \bar{x} \prod_{n \geq 1}(\bar{v}_n \hat{x}_n)$

   where $\mu_w = \mu_{w'}$, x L-spinned letter, $(v_n)_{n \geq 1}$ x-free L-spinned words, $(\breve{x}_n)_{n \geq 1}$, $(\hat{x}_n)_{n \geq 1} \in \{x, \bar{x}\}^+$ s.t. for all $n \geq 1$, $|\breve{x}_n| = |\hat{x}_n|$ and $|\breve{x}_n|_x = |\hat{x}_n|_x$;

Example

$\Delta_1 = a(bc\bar{a})^\omega$
$\Delta_2 = \bar{a}(\bar{b}\bar{c}\bar{a})^\omega$

$\Delta_1 = w\ \bar{a}\bar{b}\bar{c}\bar{a}a\bar{c}\bar{b}\bar{b}\bar{a}acab\bar{a}\ldots$
$\Delta_2 = w' abca\bar{a}cbb\bar{a}acab\bar{a}\ldots$

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3 $\Delta_1 = w\mathbf{x}$ and $\Delta_2 = w'\mathbf{y}$ where w, w' are spinned words, x and y are letters, and $\mathbf{x} \in \{x, \bar{x}\}^\omega$, $\mathbf{y} \in \{y, \bar{y}\}^\omega$ are spinned infinite words such that $\mu_w(x) = \mu_{w'}(y)$.

$\longrightarrow$ The periodic case

Which episturmian words have a unique directive word?

- A word is episturmian if and only if it can be infinitely decomposed over $\mathcal{L}_A \cup \mathcal{R}_A$.

In general, this morphic decomposition is not unique.

- Question: can we distinguish one particular (uniquely determined) morphic decomposition of each episturmian word amongst all possibilities ?
- Sturmian case

Theorem [Berthé, Holton, Zamboni 2003]

Any Sturmian word has a unique morphic decomposition containing infinitely many elements in $\{L_a, L_b\}$ but no factor of the form $R_a R_b^n L_a$ or $R_b R_a^n L_b$ with n an integer.

Which episturmian words have a unique directive word?
Normalization

- Proposition: Any episturmian word has a directive word in which appear infinitely many elements of spin L.

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- Normalization: we transform every $\bar{a}\bar{A}^*a$.

Example

$$\begin{aligned} & \bar{a}\bar{b}c\bar{b}\bar{a}\bar{c}\bar{b}\bar{a}\bar{c}a \\ = \; & \bar{a}\bar{b}c\bar{b}\bar{a}\bar{c}\bar{b}ac\bar{a} \\ = \; & \bar{a}\bar{b}c\bar{b}acb\bar{a}c\bar{a} \end{aligned}$$

Characterization

GLR2008

An episturmian word has a unique directive word if and only if its (normalized) directive word contains
1) infinitely many L-spinned letters,
2) infinitely many R-spinned letters,
3) no factor in $\bigcup_{a\in\mathcal{A}} \bar{a}\bar{\mathcal{A}}^* a$,
4) no factor in $\bigcup_{a\in\mathcal{A}} a\mathcal{A}^* \bar{a}$.
Such an episturmian word is necessarily aperiodic.

Conclusion

- Study of directive words of episturmian words
- We completed answers about words directing a common episturmian word
  - we characterized these words in an "easy-to-check" way
- Consequences:
  - normalization of directive words
  - characterization of episturmian words having a unique directive word
- Normalization can be useful to characterize quasiperiodic episturmian words.
- Other applications?

Thank you for your attention!