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# Directive words of episturmian words (Extended abstract*)

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**Abstract**

Episturmian morphisms constitute a powerful tool to study episturmian words. Indeed, any episturmian word can be infinitely decomposed over the set of pure episturmian morphisms. Thus, an episturmian word can be defined by one of its morphic decompositions or, equivalently, by a certain directive word. Here we characterize pairs of words directing a common episturmian word. As a consequence, we characterize episturmian words having a unique directive word.

**Keywords**: episturmian word; Sturmian word; Arnoux-Rauzy sequence; episturmian morphism; directive word.

MSC (2000): 68R15.

## 1 Introduction

Since the seminal works of Morse and Hedlund [21], *Sturmian words* have been widely studied and their beautiful properties are related to many fields like Number Theory, Geometry, Dynamical Systems, and Combinatorics on Words (see [1, 20, 23, 3] for recent surveys). These infinite words, which are defined on a binary alphabet, have numerous equivalent definitions and characterizations. Nowadays most works deal with generalizations of Sturmian words to arbitrary finite alphabets. Two

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very interesting generalizations are very close: the *Arnoux-Rauzy sequences* (e.g., see [2, 14, 23, 30]) and *episturmian words* (e.g., see [5, 13, 15]). The first of these two families is a particular subclass of the second one. More precisely, the family of episturmian words is composed of the Arnoux-Rauzy sequences, images of the Arnoux-Rauzy sequences by *episturmian morphisms*, and certain periodic infinite words. In the binary case, Arnoux-Rauzy sequences are exactly the Sturmian words whereas episturmian words include all recurrent *balanced* words, that is, periodic balanced words and Sturmian words (see [10, 22, 29] for recent results relating episturmian words to the balanced property). See also [9] for a recent survey on episturmian theory.

Episturmian morphisms play a central role in the study of these words. Introduced first as a generalization of Sturmian morphisms, Justin and Pirillo [13] showed that they are exactly the morphisms that preserve the aperiodic episturmian words. They also proved that any episturmian word is the image of another episturmian word by some so-called *pure episturmian morphism*. Even more, any episturmian word can be infinitely decomposed over the set of pure episturmian morphisms. This last property allows an episturmian word to be defined by one of its morphic decompositions or, equivalently, by a certain *directive word*, which is an infinite sequence of rules for decomposing the given episturmian word by morphisms. In consequence, many properties of episturmian words can be deduced from properties of episturmian morphisms. This approach is used for instance in [4, 8, 16, 28, 29, 30] and of course in the papers of Justin *et al.* In Section 2, we recall useful results on episturmian words and explain the vision of morphic decompositions and directive words introduced by Justin and Pirillo in [13].

An episturmian word can have several directive words. The question: "When do two words direct a common episturmian word?" was considered in [15]. Using a block-equivalence notion for directive words, Justin and Pirillo provided several results to answer this question in most cases (see Section 3). In Section 4, we state a complete result characterizing the form of words directing a common episturmian word, without using block-equivalence.

In Section 5, we mention two consequences of the previous characterization. The first one is a new proof of a normalization of directive words of episturmian words introduced in [17] by the second and third authors (extending a result of Berthé, Holton, and Zamboni [4]). The second one is a characterization of episturmian words having a unique directive word.

# 2 Episturmian words and morphisms

We assume the reader is familiar with combinatorics on words and morphisms (e.g., see [19, 20]). In this section, we recall some basic definitions and properties relating to episturmian words which are needed later in the paper. For the most part, we follow the notation and terminology of [5, 13, 15, 10].

## 2.1 Notation and terminology

Let $\mathcal{A}$ denote a finite *alphabet*. A finite *word* over $\mathcal{A}$ is a finite sequence of letters from $\mathcal{A}$. The *empty word* $\varepsilon$ is the empty sequence. Under the operation of concatenation, the set $\mathcal{A}^*$ of all finite words over $\mathcal{A}$ is a *free monoid* with identity element $\varepsilon$ and set of generators $\mathcal{A}$. The set of *non-empty* words over $\mathcal{A}$ is the *free semigroup* $\mathcal{A}^+ = \mathcal{A}^* \setminus \{\varepsilon\}$.

Given a finite word $w = x_1x_2\cdots x_m \in \mathcal{A}^+$ with each $x_i \in \mathcal{A}$, the *length* of $w$ is $|w| = m$. The length of the empty word is 0. By $|w|_a$ we denote the number of occurrences of the letter $a$ in the word $w$. If $|w|_a = 0$, then $w$ is said to be *a-free*. For any integer $p \geq 1$, the $p$-th power of $w$ is the word $w^p$ obtained by concatenating $p$ occurrences of $w$.

A (right) *infinite word* $\mathbf{x}$ is a sequence indexed by $\mathbb{N}^+$ with values in $\mathcal{A}$, i.e., $\mathbf{x} = x_1x_2x_3\cdots$ with each $x_i \in \mathcal{A}$. The set of all infinite words over $\mathcal{A}$ is denoted by $\mathcal{A}^\omega$. Given a non-empty finite word $v$, we denote by $v^\omega$ the infinite word obtained by concatenating $v$ with itself infinitely many times. For easier reading, infinite words are hereafter typed in boldface to distinguish them from finite words.

Given a set $X$ of words, $X^*$ (resp. $X^\omega$) is the set of all finite (resp. infinite) words that can be obtained by concatenating words of $X$. The empty word $\varepsilon$ belongs to $X^*$.

A finite word $w$ is a *factor* of a finite or infinite word $z$ if $z = uwv$ for some words $u$, $v$ (where $v$ is infinite iff $z$ is infinite). Further, $w$ is called a *prefix* (resp. *suffix*) of $z$ if $u = \varepsilon$ (resp. $v = \varepsilon$). We use the notation $p^{-1}w$ (resp. $ws^{-1}$) to indicate the removal of a prefix $p$ (resp. suffix $s$) of the word $w$.

The *alphabet* of a word $w$, denoted by $\mathrm{Alph}(w)$ is the set of letters occurring in $w$, and if $w$ is infinite, we denote by $\mathrm{Ult}(w)$ the set of all letters occurring infinitely often in $w$.

## 2.2 Episturmian words

In this paper, our vision of episturmian words will be the characteristic property stated in Theorem 2.1 below. Nevertheless, to give an idea of what an episturmian word is, let us give one of the equivalent definitions of an episturmian word provided in [5]. Before doing so, we recall that a factor $u$ of an infinite word $\mathbf{w} \in \mathcal{A}^\omega$ is *right* (resp. *left*) *special* if $ua$, $ub$ (resp. $au$, $bu$) are factors of $\mathbf{w}$ for some letters $a$, $b \in \mathcal{A}$, $a \neq b$. We recall also that the *reversal* $\widetilde{w}$ of a finite word $w$ is its mirror image: if $w = x_1 \ldots x_{m-1}x_m$, then $\widetilde{w} = x_mx_{m-1}\cdots x_1$.

An infinite word $\mathbf{t} \in \mathcal{A}^\omega$ is *episturmian* if its set of factors is closed under reversal and $\mathbf{t}$ has at most one right (or equivalently left) special factor of each length. Moreover, an episturmian word is *standard* if all of its left special factors are prefixes of it.

In the initiating paper [5], episturmian words were defined in two steps. Standard episturmian words were first introduced and studied as a generalization of standard

Sturmian words. (Note that in the rest of this paper, we refer to a standard episturmian word as an *epistandard word*, for simplicity). Then an episturmian word was defined as an infinite word having exactly the same set of factors as some epistandard word.

Moreover, it was proved in [5] that any episturmian word is *recurrent*, that is, all of its factors occur infinitely often (actually episturmian words are uniformly recurrent but this will not be needed here). An *ultimately periodic* infinite word is a word that can be written as $uv^\omega = uvvv\cdots$, for some $u, v \in \mathcal{A}^*$, $v \neq \varepsilon$. If $u = \varepsilon$, then such a word is *periodic*. Since they are recurrent, all ultimately periodic episturmian words are periodic. Let us recall that an infinite word that is not ultimately periodic is said to be *aperiodic*.

### 2.3 Episturmian morphisms

To study episturmian words, Justin and Pirillo [13] introduced *episturmian morphisms*. In particular they proved that these morphisms (defined below) are precisely the morphisms that preserve the set of aperiodic episturmian words.

Let us recall that given an alphabet $\mathcal{A}$, a *morphism* $f$ on $\mathcal{A}$ is a map from $\mathcal{A}^*$ to $\mathcal{A}^*$ such that $f(uv) = f(u)f(v)$ for any words $u$, $v$ over $\mathcal{A}$. A morphism on $\mathcal{A}$ is entirely defined by the images of letters in $\mathcal{A}$. All morphisms considered in this paper will be non-erasing: the image of any non-empty word is never empty. Hence the action of a morphism $f$ on $\mathcal{A}^*$ can be naturally extended to infinite words; that is, if $\mathbf{x} = x_1x_2x_3\cdots \in \mathcal{A}^\omega$, then $f(\mathbf{x}) = f(x_1)f(x_2)f(x_3)\cdots$.

In what follows, we will denote the composition of morphisms by juxtaposition as for concatenation of words.

Episturmian morphisms are the compositions of the permutation morphisms (the morphisms $f$ such that $f(\mathcal{A}) = \mathcal{A}$) and the morphisms $L_a$ and $R_a$ where, for all $a \in \mathcal{A}$:

$$L_a : \begin{cases} a & \mapsto & a \\ b & \mapsto & ab \end{cases}, \quad R_a : \begin{cases} a & \mapsto & a \\ b & \mapsto & ba \end{cases} \quad \text{for all } b \neq a \text{ in } \mathcal{A}.$$

Here we will work only on *pure* episturmian morphisms, i.e., morphisms obtained by composition of elements of the sets:

$$\mathcal{L}_\mathcal{A} = \{L_a \mid a \in \mathcal{A}\} \quad \text{and} \quad \mathcal{R}_\mathcal{A} = \{R_a \mid a \in \mathcal{A}\}.$$

*Note.* In [13], the morphism $L_a$ (resp. $R_a$) is denoted by $\psi_a$ (resp. $\bar{\psi}_a$). We adopt the current notation to emphasize the action of $L_a$ (resp. $R_a$) when applied to a word, which consists in placing an occurrence of the letter $a$ on the $\underline{\text{l}}$eft (resp. $\underline{\text{r}}$ight) of each occurrence of any letter different from $a$.

*Epistandard morphisms* are the morphisms obtained by concatenation of morphisms in $\mathcal{L}_\mathcal{A}$ and permutations on $\mathcal{A}$. Likewise, the *pure episturmian morphisms* (resp. *pure epistandard morphisms*) are the morphisms obtained by concatenation of morphisms in $\mathcal{L}_\mathcal{A} \cup \mathcal{R}_\mathcal{A}$ (resp. in $\mathcal{L}_\mathcal{A}$). Note that the episturmian morphisms are exactly the *Sturmian morphisms* when $\mathcal{A}$ is a 2-letter alphabet.

All episturmian morphisms are injective on both finite and infinite words. The monoid of episturmian morphisms is *left cancellative* (see [26, Lem. 7.2]) which means that for any episturmian morphisms $f, g, h$, if $fg = fh$ then $g = h$. Note that this fact, which is a by-product of the injectivity, can also be seen as a consequence of the invertibility of these morphisms (see [7, 12, 26, 32]).

## 2.4 Morphic decomposition of episturmian words

Justin and Pirillo [13] proved the following insightful characterizations of epistandard and episturmian words (see Theorem 2.1 below), which show that any episturmian word can be *infinitely decomposed* over the set of pure episturmian morphisms.

The statement of Theorem 2.1 needs some extra definitions and notation.

First we define the following new alphabet, $\bar{\mathcal{A}} = \{\bar{x} \mid x \in \mathcal{A}\}$. A letter $\bar{x}$ is considered to be $x$ with *spin* $R$, whilst $x$ itself has spin $L$. A finite or infinite word over $\mathcal{A} \cup \bar{\mathcal{A}}$ is called a *spinned* word. To ease the reading, we sometimes call a letter with spin $L$ (resp. spin $R$) an $L$-spinned (resp. $R$-spinned) letter. By extension, an $L$-spinned (resp. $R$-spinned) word is a word having only letters with spin $L$ (resp. spin $R$).

The *opposite* $\bar{w}$ of a finite or infinite spinned word $w$ is obtained from $w$ by exchanging all spins in $w$. For instance, if $v = ab\bar{a}$, then $\bar{v} = \bar{a}\bar{b}a$. When $v \in \mathcal{A}^+$, then its opposite $\bar{v} \in \bar{\mathcal{A}}^+$ is an $R$-spinned word and we set $\bar{\varepsilon} = \varepsilon$. Note that, given a finite or infinite word $w = w_1 w_2 \ldots$ over $\mathcal{A}$, we sometimes denote $\breve{w} = \breve{w}_1 \breve{w}_2 \cdots$ any spinned word such that $\breve{w}_i = w_i$ if $\breve{w}_i$ has spin $L$ and $\breve{w}_i = \bar{w}_i$ if $\breve{w}_i$ has spin $R$. Such a word $\breve{w}$ is called a *spinned version* of $w$.

*Note.* In Justin and Pirillo's original papers, spins are 0 and 1 instead of $L$ and $R$. It is convenient here to change this vision of the spins because of the relationship with episturmian morphisms, which we now recall.

For $a \in \mathcal{A}$, let $\mu_a = L_a$ and $\mu_{\bar{a}} = R_a$. This operator $\mu$ can be naturally extended (as done in [13]) to a morphism from the free monoid $(\mathcal{A} \cup \bar{\mathcal{A}})^*$ to a pure episturmian morphism: for a spinned finite word $\breve{w} = \breve{w}_1 \ldots \breve{w}_n$ over $\mathcal{A} \cup \bar{\mathcal{A}}$, $\mu_{\breve{w}} = \mu_{\breve{w}_1} \ldots \mu_{\breve{w}_n}$ ($\mu_\varepsilon$ is the identity morphism). We say that the word $w$ *directs* or is a *directive word* of the morphism $\mu_w$. The following result extends the notion of directive words to infinite episturmian words.

**Theorem 2.1.** [13]

*i) An infinite word* $\mathbf{s} \in \mathcal{A}^\omega$ *is epistandard if and only if there exist an infinite word* $\Delta = x_1 x_2 x_3 \cdots$ *over* $\mathcal{A}$ *and an infinite sequence* $(\mathbf{s}^{(n)})_{n \geq 0}$ *of infinite words such that* $\mathbf{s}^{(0)} = \mathbf{s}$ *and for all* $n \geq 1$, $\mathbf{s}^{(n-1)} = L_{x_n}(\mathbf{s}^{(n)})$.

*ii)* *An infinite word* $\mathbf{t} \in \mathcal{A}^\omega$ *is episturmian if and only if there exist a spinned infinite word* $\breve{\Delta} = \breve{x}_1\breve{x}_2\breve{x}_3\cdots$ *over* $\mathcal{A} \cup \bar{\mathcal{A}}$ *and an infinite sequence* $(\mathbf{t}^{(n)})_{n\geq 0}$ *of recurrent infinite words such that* $\mathbf{t}^{(0)} = \mathbf{t}$ *and for all* $n \geq 1$, $\mathbf{t}^{(n-1)} = \mu_{\breve{x}_n}(\mathbf{t}^{(n)})$.

For any epistandard word (resp. episturmian word) $\mathbf{t}$ and $L$-spinned infinite word $\Delta$ (resp. spinned infinite word $\breve{\Delta}$) satisfying the conditions of the above theorem, we say that $\Delta$ (resp. $\breve{\Delta}$) is a *(spinned) directive word* for $\mathbf{t}$ or that $\mathbf{t}$ is *directed by* $\Delta$ (resp. $\breve{\Delta}$). Notice that this directive word is exactly the one that arises from the equivalent definition of epistandard words that uses *palindromic closure* [5, 9, 13] and, in the binary case, it is related to the continued fraction of the slope of the straight line represented by a standard word (see [20]). It follows immediately from Theorem 2.1 that, with the notation of case *ii*), each $\mathbf{t}^{(n)}$ is an episturmian word directed by $\breve{x}_{n+1}\breve{x}_{n+2}\cdots$

The natural question: "Does any spinned infinite word direct a unique episturmian word?" is answered in [13]:

**Proposition 2.2.** [13, Prop. 3.11]

1. *Any spinned infinite word* $\breve{\Delta}$ *having infinitely many L-spinned letters directs a unique episturmian word beginning with the left-most letter having spin L in* $\breve{\Delta}$.
2. *Any R-spinned infinite word* $\breve{\Delta}$ *directs exactly* $|\mathrm{Ult}(\Delta)|$ *episturmian words.*
3. *Let* $\breve{\Delta}$ *be an R-spinned infinite word, and let a be a letter such that* $\bar{a} \in \mathrm{Ult}(\breve{\Delta})$. *Then* $\breve{\Delta}$ *directs exactly one episturmian word starting with a.*

*Note.* In [13], item 3 was stated in the more general case where $\breve{\Delta}$ is ultimately $R$-spinned. In this case, $\breve{\Delta}$ still directs exactly one episturmian word for each letter $\bar{a}$ in $\mathrm{Ult}(\breve{\Delta})$, but contrary to what is written in [13], nothing can be said on its first letter.

As a consequence of the previous proposition and part *i*) of Theorem 2.1, any $L$-spinned infinite word directs a unique epistandard word. The following important remark links the two parts of Theorem 2.1.

**Remark 2.3.** [13] If $\breve{\Delta}$ is a spinned version of an $L$-spinned word $\Delta$ and if $\mathbf{t}$ is an episturmian word directed by $\breve{\Delta}$, then the set of factors of $\mathbf{t}$ is exactly the set of factors of the epistandard word $\mathbf{s}$ directed by $\Delta$.

Moreover (with the same notation as in the previous remark):

**Remark 2.4.** The episturmian word $\mathbf{t}$ is periodic if and only if the epistandard word $\mathbf{s}$ is periodic, and this holds if and only if there is only one letter occurring infinitely often in $\Delta$, that is, $|\mathrm{Ult}(\Delta)| = 1$ (see [13, Prop. 2.9]). More precisely, a periodic episturmian word takes the form $(\mu_{\breve{w}}(x))^\omega$ for some finite spinned word $\breve{w}$ and letter $x$.

*Note.* Sturmian words are precisely the aperiodic episturmian words on a 2-letter alphabet.

When an episturmian word is aperiodic, we have the following fundamental link between the words $(\mathbf{t}^{(n)})_{n\geq 0}$ and the spinned infinite word $\breve{\Delta}$ occurring in Theorem 2.1: if $a_n$ is the first letter of $\mathbf{t}^{(n)}$, then $\mu_{\breve{x}_1\ldots\breve{x}_n}(a_n)$ is a prefix of $\mathbf{t}$ and the sequence $(\mu_{\breve{x}_1\ldots\breve{x}_n}(a_n))_{n\geq 1}$ is not ultimately constant (since $\breve{\Delta}$ is not ultimately constant), then $\mathbf{t} = \lim_{n\to\infty} \mu_{\breve{x}_1\cdots\breve{x}_n}(a_n)$. This fact is a slight generalization of a result of Risley and Zamboni [30, Prop. III.7] on *S-adic representations* for characteristic Arnoux-Rauzy sequences. See also the recent paper [4] for S-adic representations of Sturmian words. Note that *S-adic dynamical systems* were introduced by Ferenczi [6] as *minimal dynamical systems* (e.g., see [23]) generated by a finite number of substitutions. In the case of episturmian words, the notion itself is actually a reformulation of the well-known *Rauzy rules*, as studied in [25].

To anticipate next sections, let us also observe:

**Remark 2.5.** [13] If an aperiodic episturmian word is directed by two spinned words $\Delta_1$ and $\Delta_2$, then $\Delta_1$ and $\Delta_2$ are spinned versions of a common $L$-spinned word.

This is no longer true for periodic episturmian words; for instance $ab^\omega$ and $\bar{b}a^\omega$ direct the same episturmian word $(ab)^\omega = ababab\cdots$.

# 3 Known results on directive-equivalent words

We have just seen an example of a periodic episturmian word that is directed by two different spinned infinite words. This situation holds also in the aperiodic case (see [13, 15]). For example, the *Tribonacci word* (or *Rauzy word* [24]) is directed by $(abc)^\omega$ and also by $(abc)^n\bar{a}\bar{b}\bar{c}(a\bar{b}\bar{c})^\omega$ for each $n \geq 0$, as well as infinitely many other spinned words. More generally, by [13], any epistandard word has a unique $L$-spinned directive word but also has other directive words (see also [15] and Theorem 4.1).

We now consider in detail the following two questions: When do two finite spinned words direct a common episturmian morphism? When do two spinned infinite words direct a common unique episturmian word? We say that that two finite (resp. infinite) spinned words are *directive-equivalent* words if they direct a common episturmian morphism (resp. a common episturmian word).

In Section 3.1, we recall the characterizations of directive-equivalent finite spinned words. In Section 3.2, we recall known results about directive-equivalent infinite words. Section 4 will present a new characterization of these words.

## 3.1 Finite directive-equivalent words: presentation versus block-equivalence

Generalizing a study of the monoid of Sturmian morphisms by Séébold [31], the third author [26] answered the question: "When do two spinned finite words direct a common episturmian morphism?" by giving a presentation of the monoid of episturmian morphisms. This result was reformulated in [27] using another set of generators and it was independently and differently treated in [15]. As a direct consequence, one can see that the monoid of pure epistandard morphisms is a free monoid and one can obtain the following presentation of the monoid of pure episturmian morphisms:

**Theorem 3.1.** (direct consequence of [27, Prop. 6.5]; reformulation of [15, Th. 2.2])
*The monoid of pure episturmian morphisms with $\{L_\alpha, R_\alpha \mid \alpha \in \mathcal{A}\}$ as a set of generators has the following presentation:*

$$R_{a_1} R_{a_2} \dots R_{a_k} L_{a_1} = L_{a_1} L_{a_2} \dots L_{a_k} R_{a_1}$$

*where $k \geq 1$ is an integer and $a_1, \dots, a_k \in \mathcal{A}$ with $a_1 \neq a_i$ for all $i$, $2 \leq i \leq k$.*

This result means that two different compositions of morphisms in $\mathcal{L}_\mathcal{A} \cup \mathcal{R}_\mathcal{A}$ yield a common pure episturmian morphism if and only if one composition can be deduced from the other one in a rewriting system, called the *block-equivalence* in [15]. Although Theorem 3.1 allows us to show that many properties of episturmian words are linked to properties of episturmian morphisms, it will be convenient for us to have in mind the block-equivalence that we now recall.

A word of the form $xvx$, where $x \in \mathcal{A}$ and $v \in (\mathcal{A} \setminus \{x\})^*$, is called a ($x$-based) *block*. A ($x$-based) *block-transformation* is the replacement in a spinned word of an occurrence of $xv\bar{x}$ (where $xvx$ is a block) by $\bar{x}\bar{v}x$ or vice-versa. Two finite spinned words $w$, $w'$ are said to be *block-equivalent* if we can pass from one to the other by a (possibly empty) chain of block-transformations, in which case we write $w \equiv w'$. For example, $\bar{b}\bar{a}bc\bar{b}\bar{a}\bar{c}$ and $babc\bar{b}\bar{a}\bar{c}$ are block-equivalent because $\bar{b}\bar{a}b\bar{c}b\bar{a}\bar{c} \to ba\bar{b}\bar{c}b\bar{a}\bar{c} \to babc\bar{b}\bar{a}\bar{c}$ and vice-versa. The block-equivalence is an equivalence relation over spinned words, and moreover one can observe that if $w \equiv w'$ then $w$ and $w'$ are spinned versions of a common word over $\mathcal{A}$.

Theorem 3.1 can be reformulated in terms of block-equivalence:

**Theorem 3.1.** *Let $w$, $w'$ be two spinned words over $\mathcal{A} \cup \bar{\mathcal{A}}$. Then $\mu_w = \mu_{w'}$ if and only if $w \equiv w'$.*

## 3.2 Infinite directive-equivalent words: previous results

The question: "When do two spinned infinite words direct a common unique episturmian word?" was tackled by Justin and Pirillo in [15] for *bi-infinite episturmian words*, that is, episturmian words with letters indexed by $\mathbb{Z}$ (and not by $\mathbb{N}$ as considered until now). Let us recall relations between right-infinite episturmian words and bi-infinite episturmian words (see [15, p. 332] and [9] for more details).

First we observe that a right-infinite episturmian word $\mathbf{t}$ can be prolonged infinitely to the left with the same set of factors. Note also that the definition of episturmian words considered in Section 2.2 (using reversal and special factors) can be extended to bi-infinite words (see [15]). Furthermore, the characterization (Theorem 2.1) of right-infinite episturmian words by a sequence $(\mathbf{t}^{(i)})_{i\geq 0}$ extends to bi-infinite episturmian words, with all the $\mathbf{t}^{(i)}$ now bi-infinite episturmian words. That is, as for right-infinite episturmian words, we have bi-infinite words of the form $\mathbf{l}^{(i)}.\mathbf{r}^{(i)}$ where $\mathbf{l}^{(i)}$ is a left-infinite episturmian word and $\mathbf{r}^{(i)}$ is a right-infinite episturmian word. Moreover, if the bi-infinite episturmian word $\mathbf{b} = \mathbf{l}.\mathbf{r}$ is directed by $\breve{\Delta}$ with associated bi-infinite episturmian words $\mathbf{b}^{(i)} = \mathbf{l}^{(i)}.\mathbf{r}^{(i)}$, then $\mathbf{r}$ is directed by $\breve{\Delta}$ with associated right-infinite episturmian words $\mathbf{r}^{(i)}$.

As a consequence of what precedes, Justin and Pirillo's results about spinned words directing a common bi-infinite episturmian word are still valid for words directing a common (right-infinite) episturmian word. We summarize now these results, which will be helpful for the proof of our main theorem (Theorem 4.1, to follow).

First of all, Justin and Pirillo characterized pairs of words directing a common episturmian word in the case of *wavy* directive words, that is, spinned infinite words containing infinitely many $L$-spinned letters and infinitely many $R$-spinned letters. This characterization uses the following extension of the block-equivalence $\equiv$ for infinite words.

Let $\Delta_1$, $\Delta_2$ be spinned versions of $\Delta$. We write $\Delta_1 \rightsquigarrow \Delta_2$ if there exist infinitely many prefixes $f_i$ of $\Delta_1$ and $g_i$ of $\Delta_2$ with the $g_i$ of strictly increasing lengths, and such that, for all $i$, $|g_i| \leq |f_i|$ and $f_i \equiv g_i c_i$ for a suitable spinned word $c_i$. Infinite words $\Delta_1$ and $\Delta_2$ are said to be *block-equivalent* (denoted by $\Delta_1 \equiv \Delta_2$) if $\Delta_1 \rightsquigarrow \Delta_2$ and $\Delta_2 \rightsquigarrow \Delta_1$.

**Theorem 3.2.** [15, Th. 3.4, Cor. 3.5] *Let $\Delta_1$ and $\Delta_2$ be wavy spinned versions of $\Delta \in A^\omega$ with $|\mathrm{Ult}(\Delta)| > 1$. Then $\Delta_1$ and $\Delta_2$ direct a common (unique) episturmian word if and only if $\Delta_1 \equiv \Delta_2$.*

*Moreover when $\Delta_1$ and $\Delta_2$ do not have any common prefix modulo $\equiv$, and when there exists a letter $x$ such that $\Delta_1$ and $\Delta_2$ begin with $x$ and $\bar{x}$ respectively, if $\Delta_1 \equiv \Delta_2$, then $\Delta_1 = x\prod_{n\geq 1} v_n\breve{x}_n$, $\Delta_2 = \bar{x}\prod_{n\geq 1}\bar{v}_n\hat{x}_n$ for an $L$-spinned letter $x$, a sequence $(v_n)_{n\geq 1}$ of $x$-free $L$-spinned words, and sequences of spinned letters $(\breve{x}_n)_{n\geq 1}$, $(\hat{x}_n)_{n\geq 1}$ in $\{x, \bar{x}\}$ such that $(\breve{x}_n)_{n\geq 1}$ contains infinitely many times the $R$-spinned letter $\bar{x}$, and $(\hat{x}_n)_{n\geq 1}$ contains infinitely many times the $L$-spinned letter $x$.*

The relation $\rightsquigarrow$ (and hence the block-equivalence $\equiv$ for infinite words) is rather intricate to understand. So in some way the forms of $\Delta_1$ and $\Delta_2$ at the end of Theorem 3.2 are, although technical, easier to understand. Theorem 4.1, which refines the end of the previous result and proves the converse, describes all possible forms for pairs of directive-equivalent words without any use of notations $\rightsquigarrow$ and $\equiv$.

When one of the two considered directive words is not wavy, Justin and Pirillo established:

**Proposition 3.3.** [15, Prop. 3.6] *Let $\Delta_1$ and $\Delta_2$ be spinned versions of a common word such that $\Delta_1$ is wavy and letters of $\Delta_2$ are ultimately of spin $L$ (resp. ultimately* *of spin $R$). If $\Delta_1$ and $\Delta_2$ are directive-equivalent, then $\Delta_1 \rightsquigarrow \Delta_2$. Moreover there exist spinned words $w_1$, $w_2$, an $L$-spinned letter $x$, and $L$-spinned $x$-free words $(v_i)_{i\geq 1}$ such that $\mu_{w_1} = \mu_{w_2}$, $\Delta_1 = w_1\bar{x}\prod_{i\geq 1}\bar{v}_i x$ and $\Delta_2 = w_2 x\prod_{i\geq 1} v_i x$ (resp. $\Delta_1 = w_1 x\prod_{i\geq 1} v_i\bar{x}$ and $\Delta_2 = w_2\bar{x}\prod_{i\geq 1}\bar{v}_i\bar{x}$).*

With the next two results, they considered the remaining cases of words directing aperiodic episturmian words. In the first one, the spins of the letters in each of the two directive words are ultimately $L$ or ultimately $R$. The second result shows that if one of the directive words has the spins of its letters ultimately $L$ (resp. ultimately $R$), then the other directive word cannot have the spins of its letters ultimately $R$ (resp. ultimately $L$).

**Proposition 3.4.** [15, Prop. 3.7] *Let $\Delta_1$ and $\Delta_2$ be spinned versions of a common word $\Delta \in \mathcal{A}^\omega$ with $|\mathrm{Ult}(\Delta)| > 1$. If there exist spinned words $w_1, w_2$ and an $L$-spinned infinite word $\Delta'$ such that $\Delta_1 = w_1\Delta'$ and $\Delta_2 = w_2\Delta'$ (resp. $\Delta_1 = w_1\bar{\Delta}'$ and $\Delta_2 = w_2\bar{\Delta}'$), then $\Delta_1$, $\Delta_2$ are directive-equivalent if and only if $\mu_{w_1} = \mu_{w_2}$.*

**Proposition 3.5.** [15, Prop. 3.9] *Let $\Delta$ be an $L$-spinned infinite word. Then $\Delta$ and $\bar{\Delta}$ do not direct a common right-infinite episturmian word.*

Actually the previous statement is a corollary of Proposition 3.9 in [15] which considers more generally words directing episturmian words differing only by a shift.

Justin and Pirillo also discussed in [15] the periodic case and proved:

**Proposition 3.6.** [15, Prop. 3.10] *Suppose that $\Delta_1 = \breve{w}\breve{y}a^\omega$ and $\Delta_2 = \hat{w}\hat{y}\bar{a}^\omega$, where $\breve{w}$ and $\hat{w}$ (resp. $\breve{y}$ and $\hat{y}$) are spinned versions of a common word and $a$ is an $L$-spinned letter. Then $\Delta_1$ and $\Delta_2$ are directive-equivalent if and only if there exist sequences of letters $(\breve{a}_n)_{n\geq 1}$ and $(\hat{a}_n)_{n\geq 1}$ such that $\breve{w}\breve{y}\prod_{n\geq 1}\breve{a}_n \equiv \hat{w}\hat{y}\prod_{n\geq 1}\hat{a}_n$.*

We will see in Theorem 4.1 that other cases can occur for periodic episturmian words.

# 4 Directive-equivalent words: a characterization

As shown in the previous section, Justin and Pirillo provided quite complete results about directive-equivalent infinite words. Nevertheless they did not systematically provide the relative forms of two directive-equivalent words. The following characterization does it, moreover without the use of relations $\rightsquigarrow$ and $\equiv$. This result also fully solves the periodic case, which was only partially solved in [15].

**Theorem 4.1.** *Given two spinned infinite words $\Delta_1$ and $\Delta_2$, the following assertions are equivalent.*

*i) $\Delta_1$ and $\Delta_2$ direct a common right-infinite episturmian word;*

*ii)* $\Delta_1$ *and* $\Delta_2$ *direct a common bi-infinite episturmian word;*

*iii) One of the following cases holds for some* $i, j$ *such that* $\{i, j\} = \{1, 2\}$*:*

1. $\Delta_i = \prod_{n\geq 1} v_n$*,* $\Delta_j = \prod_{n\geq 1} z_n$ *where* $(v_n)_{n\geq 1}, (z_n)_{n\geq 1}$ *are spinned words such that* $\mu_{v_n} = \mu_{z_n}$ *for all* $n \geq 1$*;*
2. $\Delta_i = wx \prod_{n\geq 1} v_n \breve{x}_n$*,* $\Delta_j = w'\bar{x} \prod_{n\geq 1} \bar{v}_n \hat{x}_n$ *where* $w$*,* $w'$ *are spinned words such that* $\mu_w = \mu_{w'}$*,* $x$ *is an L-spinned letter,* $(v_n)_{n\geq 1}$ *is a sequence of non-empty* $x$*-free L-spinned words, and* $(\breve{x}_n)_{n\geq 1}$*,* $(\hat{x}_n)_{n\geq 1}$ *are sequences of non-empty spinned words over* $\{x, \bar{x}\}$ *such that, for all* $n \geq 1$*,* $|\breve{x}_n| = |\hat{x}_n|$ *and* $|\breve{x}_n|_x = |\hat{x}_n|_x$*;*
3. $\Delta_1 = w\mathbf{x}$ *and* $\Delta_2 = w'\mathbf{y}$ *where* $w$*,* $w'$ *are spinned words,* $x$ *and* $y$ *are letters, and* $\mathbf{x} \in \{x, \bar{x}\}^\omega$*,* $\mathbf{y} \in \{y, \bar{y}\}^\omega$ *are spinned infinite words such that* $\mu_w(x) = \mu_{w'}(y)$*.*

*Note.* For $a, b, c$ three different letters in $\mathcal{A}$, the spinned infinite words $\Delta_1 = a(bc\bar{a})^\omega$ and $\Delta_2 = \bar{a}(\bar{b}\bar{c}\bar{a})^\omega$ direct a common episturmian word that starts with the letter $a$. Indeed, these two directive words fulfill item 2 of Theorem 4.1 with $w = w' = \varepsilon$, $x = a$, and for all $n$, $v_n = bc$ and $\breve{x}_n = \hat{x}_n = \bar{a}$. Moreover the fact that $\Delta_1$ starts with the $L$-spinned letter $a$ shows that the word it directs starts with $a$. Similarly $\Delta'_1 = \bar{a}b(ca\bar{b})^\omega$ and $\Delta'_2 = \bar{a}\bar{b}(\bar{c}\bar{a}\bar{b})^\omega$ direct a common episturmian word starting with the letter $b$. Since $\Delta_2 = \Delta'_2$, this shows that the relation "direct a common episturmian word" over spinned infinite word is not an equivalence relation.

Items 2 and 3 of Theorem 4.1 show that any episturmian word is directed by a spinned infinite word having infinitely many $L$-spinned letters, but also by a spinned word having both infinitely many $L$-spinned letters and infinitely many $R$-spinned letters (i.e., a wavy word). To emphasize the importance of these facts, let us recall from Proposition 2.2 that if $\breve{\Delta}$ is a spinned infinite word over $\mathcal{A} \cup \bar{\mathcal{A}}$ with infinitely many $L$-spinned letters, then there exists a unique episturmian word $\mathbf{t}$ directed by $\breve{\Delta}$. Unicity comes from the fact that the first letter of $\mathbf{t}$ is fixed by the first $L$-spinned letter in $\breve{\Delta}$.

**Remark 4.2.** In items 1 and 2 of Theorem 4.1, the two considered directive words are spinned versions of a common $L$-spinned word. This does not hold in item 3, which deals only with periodic episturmian words. This is consistent with Remark 2.5. As an example of item 3, one can consider the word $(ab)^\omega = L_a(b^\omega) = R_b(a^\omega)$ which, as already said at the end of Section 2.4, is directed by $ab^\omega$ and by $\bar{b}a^\omega$ ($L_a(b) = ab = R_b(a)$). Note also that $(ab)^\omega$ is directed by $(a\bar{b})^\omega$, underlining the fact that $x$ and $y$ can be equal in item 3 of Theorem 4.1.

**Remark 4.3.** If an episturmian word $\mathbf{t}$ has two directive words satisfying items 2 or 3, then $\mathbf{t}$ has infinitely many directive words. Indeed, if item 2 is satisfied and $\bar{x}$ occurs in $\breve{x}_p$ ($p \geq 1$), then by Theorem 3.1, $x\left(\prod_{k=1}^{p-1} v_n \breve{x}_n\right) v_p \breve{x}'_p \bar{x} \equiv$

$\bar{x}\left(\prod_{k=1}^{p-1}\bar{v}_n\breve{x}_n\right)\bar{v}_p\breve{x}'_p x$ where $\breve{x}'_p$ is such that $\breve{x}_p \equiv \bar{x}\breve{x}'_p$. Thus $\mathbf{t}$ is also directed by $w\bar{x}\left(\prod_{k=1}^{p-1}\bar{v}_n\breve{x}_n\right)\bar{v}_p\breve{x}'_p x\prod_{n\geq p+1}v_n\breve{x}_n$. Similarly if item 2 is satisfied and $x$ occurs in $\breve{x}_p$ ($p \geq 1$), then $\mathbf{t}$ is also directed by $w'x\left(\prod_{k=1}^{p-1}v_n\hat{x}_n\right)v_p\hat{x}'_p\bar{x}\prod_{n\geq p+1}\bar{v}_n\hat{x}_n$ where $\hat{x}'_p$ is such that $\hat{x}_p \equiv x\hat{x}'_p$. If item 3 is satisfied, then $\mathbf{t}$ is periodic and directed by $w\mathbf{x}$ where $\mathbf{x}$ is any spinned version of $x^\omega$.

# 5 Consequences

## 5.1 Normalized directive word of an episturmian word

In the previous section we have seen that any episturmian word $\mathbf{t}$ has a directive word with infinitely many $L$-spinned letters. To work on Sturmian words, Berthé, Holton, and Zamboni recently proved that it is always possible to choose a particular directive word:

**Theorem 5.1.** [4] *Any Sturmian word* $\mathbf{w}$ *over* $\{a,b\}$ *has a unique representation of the form*

$$\mathbf{w} = \lim_{n\to\infty} L_a^{d_1-c_1}R_a^{c_1}L_b^{d_2-c_2}R_b^{c_2}\dots L_a^{d_{2n-1}-c_{2n-1}}R_a^{c_{2n-1}}L_b^{d_{2n}-c_{2n}}R_b^{c_{2n}}(a)$$

*where* $d_k \geq c_k \geq 0$ *for all integer* $k \geq 1$, $d_k \geq 1$ *for* $k \geq 2$ *and if* $c_k = d_k$ *then* $c_{k-1} = 0$.

In other words, any Sturmian word has a unique directive word over $\{a, b, \bar{a}, \bar{b}\}$ containing infinitely many $L$-spinned letters but no factor of the form $\bar{a}\bar{b}^n a$ or $\bar{b}\bar{a}^n b$ with $n$ an integer. Actually this result is quite natural if one thinks about the presentation of the monoid of Sturmian morphisms (see [31]). Using Theorems 3.1 and 4.1, one can generalize Theorem 5.1 to episturmian words:

**Theorem 5.2.** *Any episturmian word* $\mathbf{t}$ *has a spinned directive word containing infinitely many L-spinned letters, but no factor in* $\bigcup_{a\in\mathcal{A}}\bar{a}\bar{\mathcal{A}}^*a$. *Such a directive word is unique if* $\mathbf{t}$ *is aperiodic.*

The example given in Remark 4.2 shows that unicity does not necessarily hold for periodic episturmian words. A directive word of an aperiodic episturmian word $\mathbf{t}$ with the above property is called the *normalized directive word* of $\mathbf{t}$. We extend this definition to morphisms: a finite spinned word $w$ is said to be a *normalized directive word* of the morphism $\mu_w$ if $w$ has no factor in $\bigcup_{a\in\mathcal{A}}\bar{a}\bar{\mathcal{A}}^*a$.

One can observe that, by Theorem 3.1, for any morphism in $L_a\mathcal{L}_\mathcal{A}^*R_a$, we can find another decomposition of the morphism in the set $R_a\mathcal{R}_\mathcal{A}^*L_a$. Equivalently, for any spinned word in $a\mathcal{A}^*\bar{a}$, there exists a word $w'$ in $\bar{a}\bar{\mathcal{A}}^*a$ such that $\mu_w = \mu_{w'}$. This is the main idea one can use to prove next lemma. Our (omitted) proof of Theorem 5.2 is based on an extension of this lemma to infinite words.

**Lemma 5.3.** *Any pure episturmian morphism has a unique normalized directive word.*

**Example 5.4.** Let $f$ be the pure episturmian morphism with directive word $\bar{a}\bar{b}c\bar{b}a\bar{b}\bar{a}\bar{c}\bar{b}\bar{a}\bar{c}a$. By Theorem 3.1, $\mu_{\bar{a}\bar{c}\bar{b}\bar{a}\bar{c}a} = \mu_{\bar{a}\bar{c}\bar{b}ac\bar{a}} = \mu_{acb\bar{a}c\bar{a}}$ and hence $\mu_{\bar{a}\bar{b}c\bar{b}a\bar{b}\bar{a}\bar{c}\bar{b}\bar{a}\bar{c}a} = \mu_{\bar{a}\bar{b}c\bar{b}a\bar{b}acb\bar{a}c\bar{a}}$ and $\bar{a}\bar{b}c\bar{b}a\bar{b}acb\bar{a}c\bar{a}$ is the normalized directive word of $f$.

## 5.2 Episturmian words having a unique directive word

In Section 4 we have characterized pairs of words directing a common episturmian word. In Section 5.1 we have proposed a way to uniquely define any episturmian word through a normalization of its directives words (as mentioned in the introduction, see [4, 18, 17, 11] for some uses of this normalization). These results allow to prove a characterization of episturmian words having a unique directive word.

**Theorem 5.5.** *An episturmian word has a unique directive word if and only if its (normalized) directive word contains 1) infinitely many $L$-spinned letters, 2) infinitely many $R$-spinned letters, 3) no factor in $\bigcup_{a\in\mathcal{A}} \bar{a}\bar{\mathcal{A}}^*a$, 4) no factor in $\bigcup_{a\in\mathcal{A}} a\mathcal{A}^*\bar{a}$.*

*Such an episturmian word is necessarily aperiodic.*

As an example, a particular family of episturmian words having unique directive words consists of those directed by *regular wavy words*, i.e., spinned infinite words having both infinitely many $L$-spinned letters and infinitely many $R$-spinned letters such that each letter occurs with the same spin everywhere in the directive word. More formally, a spinned version $\breve{w}$ of a finite or infinite word $w$ is said to be *regular* if, for each letter $x \in \mathrm{Alph}(w)$, all occurrences of $\breve{x}$ in $\breve{w}$ have the same spin ($L$ or $R$). For example, $a\bar{b}aa\bar{c}\bar{b}$ and $(a\bar{b}c)^\omega$ are regular, whereas $a\bar{b}a\bar{a}\bar{c}b$ and $(a\bar{b}\bar{a})^\omega$ are not regular.

In the Sturmian case, we have:

**Proposition 5.6.** *Any Sturmian word has a unique spinned directive word or infinitely many spinned directive words. Moreover, a Sturmian word has a unique directive word if and only if its (normalized) directive word is regular wavy.*

Proposition 5.6 shows a great difference between Sturmian words and episturmian words constructed over alphabets with at least three letters. Indeed, when considering words over a ternary alphabet, one can find episturmian words having exactly $m$ directive words for any $m \geq 1$. For instance, the episturmian word $\mathbf{t}$ directed by $\Delta = a(b\bar{a})^{m-1}b\bar{c}(ab\bar{c})^\omega$ has exactly $m$ directive words, namely $(\bar{a}\bar{b})^i a(b\bar{a})^j b\bar{c}(ab\bar{c})^\omega$ with $i + j = m - 1$. Notice that the suffix $b\bar{c}(ab\bar{c})^\omega$ of $\Delta$ is regular wavy, and the other $m-1$ spinned versions of $\Delta$ that also direct $\mathbf{t}$ arise from the $m-1$ words that are block-equivalent to the prefix $a(b\bar{a})^{m-1}$.

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