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Euler and his contribution to number theory

Biography

Born in 1707, Leonhard Euler was the son of a Protestant minister from the vicinity of Basel, Switzerland. With the aim of pursuing a career in theology, Euler entered the University of Basel at the age of thirteen, where he was tutored in mathematics by Johann Bernoulli (of the famous Bernoulli family of mathematicians). He developed an interest in mathematics, which consequently led him to abandon his plans to follow in his father's footsteps.

At the age of 16, Euler completed a master's degree in Philosophy and, in 1727, Peter the Great invited Euler to join the Academy of Sciences in St Petersburg, where he became a Professor of Physics in 1730, and a Professor of Mathematics in 1733 — the same year he married Catherina Gsell. In 1741, at the invitation of Frederick II, Euler became director of Mathematics and Physics at the Royal Academy of Berlin, but after a disagreement with Frederick, he returned to the Academy in St Petersburg in 1766, on invitation of Catherine the Great.

Euler was incredibly prolific, writing more than 700 books and papers. His papers accumulated at such a rapid rate that he left a pile of papers to be published by the Academy. However, they published the top papers first so that later results were published before the results they superseded or depended on. Euler's unusual productivity continued throughout the last twelve years of his life when he was entirely blind (as a consequence of not taking necessary care of himself after a cataract operation). Incredibly, after his return to St Petersburg, Euler produced almost half of his total works. Euler also had thirteen children, only five of which survived, and was able to continue his research while a child or two bounced on his knees. He has been quoted as saying that he made some of his greatest discoveries while holding a baby in his arms.

On 18 September 1783, Euler died from a brain haemorrhage not long after giving one of his grandchildren a mathematics lesson. He left so much unpublished work that the Academy did not finish publication of his work for forty-seven years after his death. The publication of the collected works and letters of Euler, the *Opera Omnia*, by the Swiss Academy of Sciences will require more than eighty-five large volumes of which seventy-six have already been published (as of late 1999).

Algebraic number theory

Euler's work on the solution to Diophantine equations (equations with integer solutions) marks the early history of algebraic number theory. A technique for finding integer solutions of equations of the form

$$z^3 = ax^2 + by^2,\ a,\ b \text{ integers},\ a > 0$$

was one particular problem considered by Euler. After developing a method of solution, he specialised by setting $a = 1$, $b = 2$, and $y = \pm 1$ to solve $z^3 = x^2 + 2$. Euler first factored this equation over the complex numbers to get:

$$z^3 = x^2 + 2 = \left(x + \sqrt{-2}\right)\left(x - \sqrt{-2}\right).$$

Now consider the set of all numbers of the form

$$\left\{a + b\sqrt{-2} \mid a,\ b \text{ integers}\right\};$$

we denote this set $\mathbf{Z}\left[\sqrt{-2}\right]$. Euler used the arithmetic properties of this set to solve the equation. (This was very bold of Euler since most mathematicians of that era were reluctant to use complex numbers in their research). He made the following two assumptions:

1. If α, β, γ are in $\mathbf{Z}\left[\sqrt{-2}\right]$ with α, β relatively prime (i.e. α, β have no common factors) in $\mathbf{Z}\left[\sqrt{-2}\right]$ and $\alpha\beta = \gamma^3$, then α, β are cubes of numbers in $\mathbf{Z}\left[\sqrt{-2}\right]$.

2. If u and v are relatively prime integers, then $u + v\sqrt{-2}$ and $u - v\sqrt{-2}$ are relatively prime in $\mathbf{Z}\left[\sqrt{-2}\right]$.

Under the above assumptions, Euler established that

$$x + \sqrt{-2} = \left(a + b\sqrt{-2}\right)^3$$

and

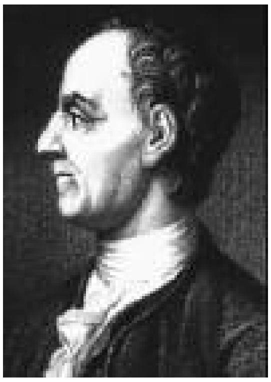
$$x - \sqrt{-2} = \left(a - b\sqrt{-2}\right)^3$$

for some integers a, b, and therefore

$$z^3 = \left(a + b\sqrt{-2}\right)^3\left(a - b\sqrt{-2}\right)^3 = \left(a^2 + 2b^2\right)^3$$

so that

$$z = a^2 + 2b^2.$$



If we expand the right-hand side of

$$x + \sqrt{-2} = \left(a + b\sqrt{-2}\right)^3$$

and equate real and imaginary parts, we get $1 = b(3a^2 - 2b^2)$. Thus, it is easily deduced that $b = 1$, $a = \pm 1$, $z = 3$, $x = \pm 5$. Using similar ideas, Euler also presented a proof of Fermat's Last Theorem for $n = 3$; that is, he claimed to have proved that there are no non-trivial integer solutions of $z^3 = x^3 + y^3$. His proof involved numbers in

$$\mathbf{Z}\left[\sqrt{-3}\right] = \left\{a + b\sqrt{-3} \mid a,\ b \text{ integers}\right\}$$

but there were problems with his approach since there were several gaps and errors in this reasoning. In fact, in this particular area of number theory, Euler did not justify many of his assumptions, and stated many claims without proof. A deeper knowledge of factorisation in algebraic number fields is required for a more complete treatment. However, this very subject is still of great research interest today.

Exercise

Show that $\mathbf{Z}\left[\sqrt{-2}\right]$ is closed under addition, subtraction, multiplication and division (not by zero). That is, if a, b belong to $\mathbf{Z}\left[\sqrt{-2}\right]$, then so do $a \pm b$, $a.b$, and a/b ($b \neq 0$).
$\mathbf{Z}\left[\sqrt{-2}\right]$ is an example of a *field*.

Analytic number theory

From the time of Fermat (1601–1665) to that of Euler, the development of analysis was of primary interest. Consequently, there was very little progress in number theory during this period. Euler not only revived number theory, but applied analytic methods to his studies. Perhaps the result that brought Euler the most fame in his earlier years was his solution to what had become known as the *Basel problem*. This was to find an expression for the sum of the infinite series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots$$

Many of the top mathematicians, including the Bernoulli family, had tried unsuccessfully to find this sum. However, in 1735, Euler succeeded where they had failed by verifying not only numerically, but exactly, the value of $\pi^2/6$ for the above sum. Euler's investigations did not stop there, and he proceeded to consider the sums of the reciprocals of higher powers. In 1736 he discovered that

$$1+\frac{1}{2^{2k}}+\frac{1}{3^{2k}}+\frac{1}{4^{2k}}+\frac{1}{5^{2k}}+\ldots=\frac{2^{2k-1}\,|\,B_{2k}\,|\,\pi^{2k}}{(2k)!}$$

where the B_{2k} are the *Bernoulli numbers* defined by

$$\frac{x}{e^x-1}=B_0+\frac{B_1x}{1!}+\frac{B_2x^2}{2!}+\frac{B_3x^3}{3!}+\ldots$$

After discovering this result on

$$\sum\frac{1}{n^{2k}}$$

Euler's attention turned to the *zeta function*:

$$\zeta(s)=1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\ldots$$

With some difficulty, he established that

$$\zeta(s)=\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{7^s}\right)\ldots$$

where the infinite product on the right is taken over all the primes 2, 3, 5, 7... It is known today as the *Euler Product Formula*.

Exercise

In the formula defining the Bernouilli numbers, substitute the series for e^x, multiply the right hand series by the series for $e^x - 1$ (first few terms, assuming the operation is valid), and hence evaluate the first few Bernouilli numbers.

Arithmetic functions

Euler was no stranger to cleaning up Fermat's incompleteness. Fermat had asserted, among other things, that:

1. the numbers of the form $2^{2^n}-1$ (the nth Fermat number) are always prime; and
2. if p is a prime and a is a positive integer not divisible by p, then $a^{p-1}-1$ is exactly divisible by p.

The first of these conjectures had simply been stated by Fermat, who left it unproven. In 1732, Euler set out to prove this result, but after several attempts he was unable to find a proof. He then tried to disprove Fermat's claim, and succeeded by finding that the fifth Fermat number, $F_5=2^{2^5}-1$ = 4 294 967 297, is divisible by 641. For the second of these conjectures, known as *Fermat's Little Theorem*, Euler published a surprisingly elementary proof in 1736, simply using mathematical induction on the natural number a. Having proved this result, Euler also established a somewhat more general statement, in which he used the now well-known *Euler phi-function*. For a positive integer n, he defined $\phi(n)$ to be the number of integers between 1 and n that are *relatively prime* to n (i.e. have no common factor with n greater than 1). For example, $\phi(4) = 2$. Clearly, if p is a prime then $\phi(p) = p - 1$, and it can be proved that

$$\phi(n)=n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\left(1-\frac{1}{p_r}\right)$$

where p_1, p_2, ... p_r are the distinct prime factors of n. Using this result, Euler generalised Fermat's Little Theorem by proving that if a and m are relatively prime, then m divides $a^{\phi(m)} - 1$. That is, if a, m are relatively prime, then $a^{\phi(m)} \equiv 1 \pmod{m}$. This is now known as Euler's Theorem, although the congruence notation was later introduced by Gauss in 1801. In addition to the phi-function, Euler also introduced the following arithmetic functions:

$d(n)$ = number of positive divisors of n;
$\sigma(n)$ = sum of the positive divisors of n.

These functions are multiplicative in the sense that if m, n are relatively prime then $f(mn) = f(m)f(n)$. If $n=p_1^{\alpha_1}.p_2^{\alpha_2}\ldots p_r^{\alpha_r}$ is the factorisation of a positive integer n into its prime powers, then by the assumption of multiplicativity, Euler showed that

$$d(n) = (\alpha_1 + 1)(\alpha_2 + 1)\ldots(\alpha_r + 1).$$

This is immediately evident from the fact that the positive divisors of p^α are 1, p, p^2, ... p^α so that $d(p^\alpha) = \alpha + 1$. It is also a simple exercise to derive

$$\sigma(n) = \left(\frac{p_1^{\alpha_1+1}-1}{p_1-1}\right)\left(\frac{p_2^{\alpha_2+1}-1}{p_2-1}\right)\cdots\left(\frac{p_r^{\alpha_r+1}-1}{p_r-1}\right).$$

Euler's memoir on $\sigma(n)$ contains many beautiful results relating to primes and partitions. The study of such arithmetic functions allowed Euler to experiment, guess results, and verify their plausibility.

Exercise

Evaluate the first few Fermat numbers. Are they prime?
Evaluate $\phi(n)$, $d(n)$, $\sigma(n)$ for the first few values of n. Can you make any observations about them?
Can you think of any other multiplicative functions?

Partitions

Many of Euler's contributions to number theory arose from his enthusiasm for interesting questions about the integers. One particular question that caught Euler's attention was raised by the Berlin mathematician, Naude. In 1740, Naude wrote to Euler to ask in how many ways a given positive integer can be expressed as a sum of r distinct positive integers. This problem was quickly solved by Euler and, within few months, he sent a memoir on the subject to the St Petersburg Academy. He had created a new area of number theory, which intrigued him for many years to follow.

First Euler introduced the idea of a partition of a positive number n into r parts as a sequence, $n_1 < n_2 < \ldots < n_r$, of positive integers such that

$$n = n_1 + n_2 + \ldots + n_r$$

where the n_i are the parts. For example, the partitions of 4 are:

$$1 + 1 + 1 + 1,\ 1 + 1 + 2,\ 1 + 3,\ 2 + 2,\ 4.$$

He then let $p(n)$ denote the number of partitions of n into any number of parts, where $p(0) = 1$. In order to study the sequence $\{p(n)\}$, he introduced the series (or *generating function*) $\sum p(n)x^n$, and showed that

$$1 + p(1)x + p(2)x^2 + \ldots = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right)\cdots$$

From this, Euler then proved that the number of partitions of n into distinct parts is the coefficient of x^n in the series for

$$(1+x+\ldots)(1+x^2+\ldots)(1+x^3+\ldots)(1+x^4+\ldots)\ldots$$

and in fact, the number of partitions of n into distinct parts equals the number of partitions of n into odd parts.

The theory of partitions was a perfect subject for Euler to exercise his great skill in formal manipulation, and he went on to prove numerous important identities. Apart from partitions, Euler also realised an even wider use for power series in number theory. He stated in a letter to Goldbach that the coefficient a_n in the series

$$\sum a_n x^n = \left(\sum x^{n^2}\right)^4$$

is the number of ways to express n as a sum of four integer squares. Thus, if it could be proved that $a_n > 0$, for all n, then Fermat's conjecture, that every positive integer is the sum of four squares, would be true. This very representation was used in the 1800s by Jacobi (1804–1851) when he used the theory of elliptic functions to prove Fermat's claim was indeed true.

Exercise

List the possible partitions of the first few integers n. Count the number of partitions for each n, and then compare this with the calculated value using Euler's formula.

References

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