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Learning mathematics through conversation

and utilizing technology ®

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Abstract

This paper discusses how students' participation in conversation and classroom activities potentially evidences and constitutes their cognition. Participation is viewed in terms of reflective discourse, a construct from the literature, and is described in the context of two Year 11 students together designing a simple aplet for their graphics calculators, then discussing its operation. Reflective discourse is characterised by shifts in conversation so that concepts which are discussed initially as resulting from mathematical operations (calculations) become referred to, in turn, as objects that are operated on, to solve problems or for developing other concepts. The aplet was for calculating the magnitude of vectors given in component form. Interaction with each other, which centred on the technology, was seen to be instrumental to the students moving from understanding magnitude in its component definition, to later using magnitude to solve vector problems in an insightful way. Using reflective discourse as a framework for analysis suggested it is a valuable theoretical viewpoint for describing how learning might occur.

Introduction

This paper was written as part of a research study into Year 11 students' learning of vectors. The research, which is ongoing, is inquiring into the roles of classroom participation and use of graphics calculators in the development of mathematical understanding. The underlying assumption of the research is that the psychological and social processes of mathematics learning are intrinsically related: "neither the cognition of individuals nor the mutually constructed network of obligations and expectations are primary; we find it impossible to give an adequate explanation of one without considering the other" (Cobb, Wood & Yackel, 1991, p. 163).

The discussion here centres on two students' design and use of a graphics calculator aplet. The analysis is framed, from the cognitive perspective, by Sfard's (1991) theory of mathematical development and, from the social perspective, by Cobb, Boufi, McClain and Whitenack's (1997) theoretical construct of reflective discourse. In Sfard's process/product view of development, students move through three stages in understanding concepts. First, a new concept is usually encountered as the result of mathematical processes, then students move to thinking of the concept as an entity separate to the processes that generated it, and finally are able to use it to solve problems. For research and teaching, knowing the stage a student is at is difficult because understanding is inherently invisible to everyone except the student him/herself. However, progress can be surmised from students' conversations, where a shift in the way a concept is referred to reflects movement from one developmental stage to the next. Classrooms where students initiate or contribute to shifts in the conversations of a group, rather than the shifts always being stated first by the teacher, and where students individually reflect on the discourse to move forward in their thinking, are said to exhibit reflective discourse (Cobb et al., 1997).

Cobb, Boufi, McClain and Whitenack (1997) name and define reflective discourse in the context of elementary schooling. In this paper the explanatory power of the theoretical construct is explored in the context of upper-secondary students' learning of vectors. Reflective discourse with its shifts defined by Sfard's (1991) developmental theory was a valuable tool for structuring the inquiry, allowed students' progress in mathematics to be tracked in a linear way, and has implications for teaching. However, the dual social/developmental perspective incumbent in reflective discourse needs to be supplemented in order to describe the complexity of learning at upper-secondary level. Adding to the description could include considering the connection of ideas while within a developmental stage (Ausubel, 1968, cited in Novak, 1978), and consideration of other modes of learning, such as writing, which evidence and constitute cognition.

Theoretical viewpoints

The ongoing study is underpinned by the theory of social constructivism (Cobb, Wood, Yackel & McNeal, 1992), where personal construction of knowledge is seen to be socially facilitated, and by radical constructivism (Noddings, 1990; von Glasersfeld, 1990), which emphasises the personal dimension of learning. Constructivism falls within what Cobb, Gravemeijer, Yackel, McClain and Whitenack (1997) call an "emergent perspective. In this approach individual thought and social and cultural approaches are considered to be reflexively related" (p. 152) but the linkage between social action and individual cognition is seen as indirect. Mutual understanding between participants is open to interpretation by all of them (Steffe, 1995), and moving from mutual to personal understanding involves a reflective process (Cobb, Boufi, McClain & Whitenack, 1997). The main referents for analysis of students' learning in the classroom episode which is described in this paper were, for the cognitive dimension, Sfard's (1991) theory of mathematical development and, for the social dimension, Cobb, Boufi, McClain and Whitenack's (1997) theory of reflective discourse, which was informed by Sfard's work.

Stages of mathematical development

Sfard (1991) describes two ways to consider mathematical ideas. First, an *operational* view where mathematical ideas are seen to encompass processes. Second, a*structural* view where mathematical ideas are viewed as objects (abstract entities) and "[t]hese objects have certain features and are subjected to certain processes governed by well defined laws" (p. 3). For example, an operational view of the displacement vector 60km in the NE direction would be to think of it, in reduced scale, as a movement in a NE direction from one point to another 60 km away (see Figure 1). A structural view could be thinking of it as a ray (as a whole), of length 60 km pointing NE.

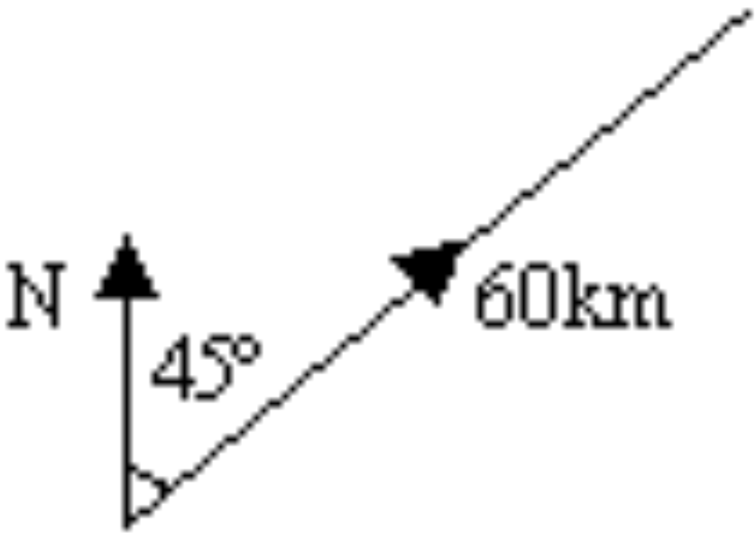
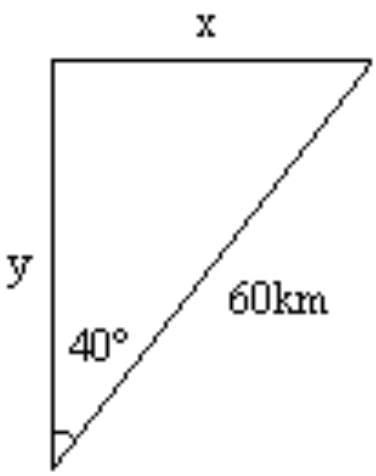


Figure 1. Displacement vector 60km in a NE direction

"[W]hereas the structural conception is static, instantaneous, and integrative, the operational is dynamic, sequential, and detailed" (p. 4). Sfard argues that both views are complementary and that "in order to speak about mathematical *objects*, we must be able to deal with the *products* of some processes without bothering about the processes themselves" (p. 10). In other words, operational conceptions precede the development of structural concepts. Sfard proposes that conceptual development, starting with the operational and moving to the structural, occurs in three stages: *interiorisation*, which involves the learner carrying out processes on familiar objects, where the processes will eventually give rise to a new object; *condensation*, where the idea incumbent in the processes starts to emerge as an autonomous entity so that the learner feels less need to go into process details; and *reification,* where the ability to see the new entity as an object is acquired and various representations of the concept become unified. Reification is evidenced by a student being able to investigate properties of the new object, solve problems where the new object satisfies given conditions, and use the new object as input into processes. These three stages in the conceptual development of the concept 'vector components' might involve the following.

1. Interiorisation: A student calculates vector components by applying trigonometry to a right triangle. For example, the hypotenuse of a right triangle is used to represent a distance in a given direction, i.e., a displacement vector; then the sine and cosine ratios are used to obtain the lengths of sides, which are then written as components (see Figure 2). Here the familiar concepts of distance and direction are operated on to yield the new concept, vector components, and at the same time new terminology and syntax is introduced.



$$\frac{x}{60} = \sin 40°$$

$$x = 60 \sin 40°$$

$$x = 38{\cdot}57 \text{ (2 dp)}$$

$$\frac{y}{60} = \cos 40°$$

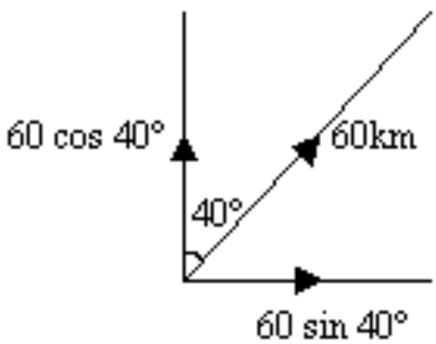
$$y = 60 \cos 40°$$

$$y = 45{\cdot}96 \text{ (2 dp)}$$

$$60\text{km at a bearing of } 040° = 38{\cdot}57i + 45{\cdot}96j$$

Figure 2. Vector components as the result of a trigonometric process.

2. Condensation: The student, when given a displacement of 60km at a bearing of 040° , can write down the components without drawing a right triangle and without carrying out the process of finding the sine and cosine ratios (see Figure 3).

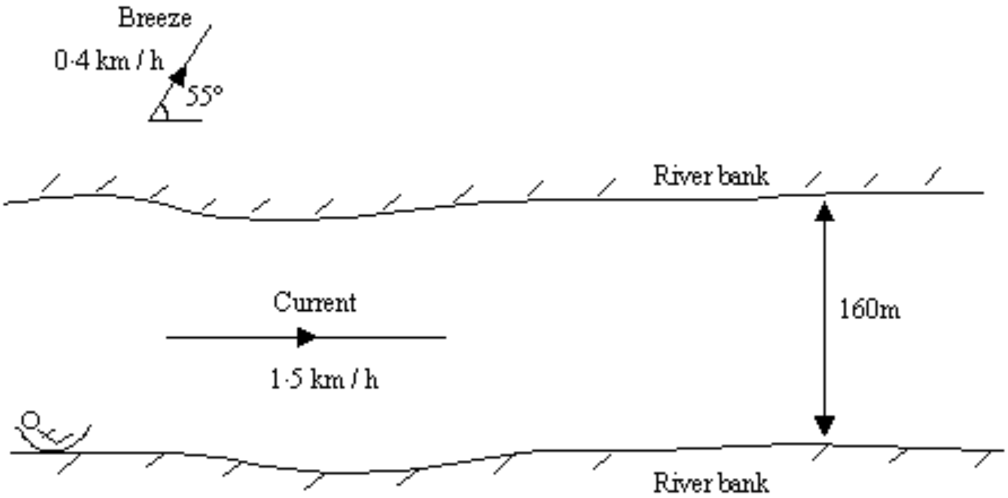


$$60 \sin 40°\, i + 60 \cos 40°\, j$$

Figure 3. Vector components as objects.

3. Reification: The student understands components as objects and operates on them to solve problems as in the following example, which was one in a set of problems designed for classroom implementation in this study:

The bible story of Moses tells us that his mother placed him in a basket among the reeds in a river. Imagine the current of the river was moving at 1.5km /h and that a gentle breeze was blowing at 0.4 km /h, at an angle of 55° to the riverbank. The river was 160m wide. The current and wind, together, carried the basket downstream for Moses to be found by the Queen of Egypt who was on the opposite bank of the river to where Moses' mother had left him. How far downstream did Moses travel before he was found? (see Figure 4)



Across the river: Time = distance/velocity

$= 0{\cdot}160/(0{\cdot}4\sin 55°)$

$= 0{\cdot}488$ h (3 dp)

Down the river: Distance = velocity x time

$= (1{\cdot}5 + 0{\cdot}4\cos 55°) \times 0{\cdot}488$

$= 0{\cdot}884$ km

Figure 4. Operating on vector components to solve a problem.

While Sfard's (1991) emphasis is psychological, on stages of development in the mind, Cobb, Boufi, McClain and Whitenack (1997) describe how social interaction can both evidence and constitute the developmental stages. They describe *shifts* in discourse where a student refers to a concept as an object instead of as the result of a process. These shifts might be evidence of reflection on prior social activity, or might be part of the process through which a student breaks new ground in an instantaneous way. "This perspective acknowledges that both the process of mathematical learning and its products, increasingly sophisticated mathematical ways of knowing, are social through and through" (p. 264). So, while Sfard (1991) theorises on the stages of mathematical development for individuals, Cobb et al. (1997) propose how the stages might be socially constituted, and provide examples to support their views in the context of first-grade students learning number concepts.

Reflective discourse

Cobb, Boufi, McClain and Whitenack (1997) characterise classrooms where students initiate shifts in public discourse as exhibiting *reflective discourse*. That is, students themselves contribute to the development of mathematical understanding of the class as a whole. While social constructivist theory assumes that knowledge is constructed by the individual through active engagement in activities (Tobin & Tippins 1993; Wood, Cobb & Yackel, 1995), for example by contributing to class discussion, the concept of reflective discourse hinges on individual students' activities being of potential benefit to other students. The benefit is realised when students reflect individually on the social action. The reflective process leading to new understanding was named by Piaget (1972) as *reflective abstraction* and involves students perceiving differences between their prior understanding and a new suggestion, then making sense of the differences.

Shifts in reflective discourse can be manifested in different ways, and Sfard's (1991) description of stages of development and movement between the stages, allows a prediction of what the nature of shifts might be. She suggests that "whereas interiorisation and condensation are gradual, quantitative rather than qualitative changes, reification is an instantaneous quantum leap" (p. 20). Shifts in discourse, therefore, might be expected to vary from being hardly noticeable when students move towards objectifying the result of a process, but marked, and more demanding, in their realising how to use the new object as input to another process. Sfard (1991) also suggests that mental pictures or visualisations "being compact and integrative, seem to support the structural conception" (p. 6), while verbal encoding "cannot be grasped 'at one glance' and must be processed sequentially, so it seems more appropriate for representing computational procedures" (p. 7). So, conversation might lean towards being centred on processes rather than involve discussion of objects on their own. In addition, the operational to structural developmental order might be reversed for geometry where the "static graphical representations appear to be more natural than any other, [so] can probably be conceived structurally even before full awareness of the alternative procedural descriptions have been applied" (Sfard, 1991, p. 10). This reversal might also apply to vectors, a topic area that has geometric as well as algebraic formulations.

In summary, this paper explores both individual and social processes of learning in the context of a Year 11 Geometry and Trigonometry course. It analyses students' learning of vectors in terms of Sfard's (1991) three stages of mathematical development and Cobb, Boufi, McClain and Whitenack's (1997) reflective discourse. These theories do not stand alone, with many other researchers including Gray and Tall (1994) and Kaput (1979) describing process/product perspectives of mathematical understanding and Voigt (1994) and Wheatley (1993) describing the social facilitation of it. While adopting the theories of Sfard (1991) and Cobb et al. (1997) as frameworks for analysis, I (first author) recognised that: "The relationship between theory and data is dialectic in that they have a tendency for generating each other. It is notable that the persuasive power of data may be confined to the paradigm within which they came to being" (Sfard, 1998, p. 12). Therefore, I sought further explanation for the learning that occurred, but mention it only briefly in this paper.

Research Methodology

Ethnomethodology (Coulon, 1995; Holstein & Gubrium, 1994; Livingston, 1987) was adopted as the research method for the part of the study that is described in this paper. This personal experience research methodology, which focuses on the concrete particulars of social interaction, has informed many studies in mathematics education. For example, it has guided inquiries into the two genders' relationships to mathematics (Jungwirth, 1996), into how mathematical meaning was constituted with students through grades one and two

(Voigt, 1994) and into students' learning of geometry (Livingston, 1987). The method is based on the assumption that meaning is "a product of social processes. . . mathematical meanings are primarily studied as emerging between individuals, not as constructed inside or as existing independently of individuals" (Voigt, 1994, p. 172). However, I don't take the 'emergence' of meaning to imply the sociocultural view that social interaction directly constitutes mathematical understanding: rather understanding is seen to develop indirectly and individually through reflection (Cobb, Boufi, McClain & Whitenack, 1997; Steffe, 1995).

Ethnomethodology, an interpretative methodology, is founded on Husserl's (1970) argument that "human consciousness constitutes the objects of experience" (cited in Holstein & Gubrium, 1994, p. 263). Explanation is sought locally--in the institutions we live, work and learn in--for any visible sense of order in how participants interact with each other, on the assumption that the order is established "from within" by all participants (Holstein & Gubrium, 1994, p. 265). The method relies on "naturally occurring talk to reveal the ways ordinary interaction produces social order in the settings where the talk occurs" (Holstein & Gubrium, 1994, p. 265) and involves seeking patterns in the text of situations.

The study took place in a private girls' college in Western Australia and involved a Year 11 class of 18 students. The teacher and I had been colleagues for ten years and I had taught for one term five of the students when I was a teacher at the college and when they were in Year 10. One of these six students, Jenny, features in the classroom episode that is described in this paper. Starting with the students' first lesson on vectors in two-dimensional space, I attended fifteen out of the seventeen fifty-minute lessons over one month on the topic. In the lessons, I took the role of participant-observer (Atkinson & Hammersley, 1994), observing whole-class work and being an assistant teacher in small-group work. I kept a journal, made brief field-notes in class, and video-taped the fifteen lessons that I attended, with the video-camera set to record continuously from a corner of the room. In addition, I audio-taped the conversations over at least ten of the fifteen lessons for each of five students. The video and audio transcripts were the primary sources of data. After transcribing at least some of the recordings the night after each lesson, I asked the teacher the next day for his opinion on points of issue, for example about students' ownership of ideas, and recorded his responses in my journal. Similarly, when I felt I needed clarification about students' taped conversations, I sought clarification from those concerned when opportunities arose during small-group work. All written work of the five students was photocopied, and students' work from three assessments, a short test question, a whole-lesson investigation, and a whole-lesson topic test was photocopied for all eighteen students in the class.

Although data generation was designed as being comprehensive, it, together with the analysis, admittedly gives only one of the many possible versions of events. However, to guard against idiosyncrasy, Guba and Lincoln's (1989) criterion of *credibility*, appropriate to interpretative research, was adopted. Credibility was achieved through persistent observation during the fifteen lessons; checking my initial impressions with the teacher and students, as described above; replaying parts of tapes to uncover selective perception in the excerpts chosen for analysis from the transcripts; searching the transcripts manually as well as electronically for shifts in discourse; writing and rewriting the episode, with each stage iteratively informing the next for progressive subjectivity; asking two past teaching colleagues to check that my portrayal of classroom life was believable; asking the teacher in the episode to read it. He had "no problems" with the interpretation.

A classroom episode

In this section of the paper, excerpts from whole-class discussion and small-group work are interpreted in terms of Sfard's (1991) theory of stages of mathematical development and

Cobb, Boufi, McClain and Whitenack's (1997) construct of reflective discourse. The focus is on two students, Jenny and Katie, who took active roles in class discussion and had a close working relationship with each other. The topic the students were studying was vectors in two-dimensional space and the discussion here relates to vector magnitude. The episode is in two parts: 'Background', followed by 'Absolute value objectified?' which describes the action and discusses it from cognitive and social perspectives. The section finishes with a summary of 'Subsequent events'.

Background

Year 11 Geometry and Trigonometry is designed for students with a "strong background in algebra and trigonometry" (Curriculum Council, 1999, p. 15) and it is studied concurrently with Introductory Calculus. Prior to the topic on vectors, which started four weeks into the school year, students had been working on trigonometry, including the sine and cosine rules, which could be used for solving vector problems. All students owned a Hewlett Packard HP38G graphics calculator with limited symbolic processing, which they had used only since the beginning of the year.

The episode below took place in the students' first lesson on vectors. Using a problem-based learning approach (Wheatley, 1993), the teacher (Mr C) started the lesson with students working in pairs on a worksheet problem to establish the meaning of vector components from the trigonometry that they knew already, in a context that was chosen to be meaningful to them. The problem had students walking to their college from a nearby bridge and solving the problem was consistent with *interiorisation* (Sfard, 1991) of 'vector' and 'vector component' concepts. The problem was the first of a set which I had designed for the vector topic for the teacher to use at his discretion. After most students had finished the problem, the teacher led class discussion on it, emphasizing the idea of signed numbers (positive and negative) in the horizontal and vertical directions, and terminology and syntax associated with vectors and vector components. Students then moved onto practice exercises from their textbook (Sadler, 1993, p. 64), involving evaluation of magnitude from vector components, the reverse operation to what had been carried out in solving the problem (see Figure 5 for a sample question with solution). The episode has Jenny and Katie programming their graphics calculators to automate their magnitude calculations.

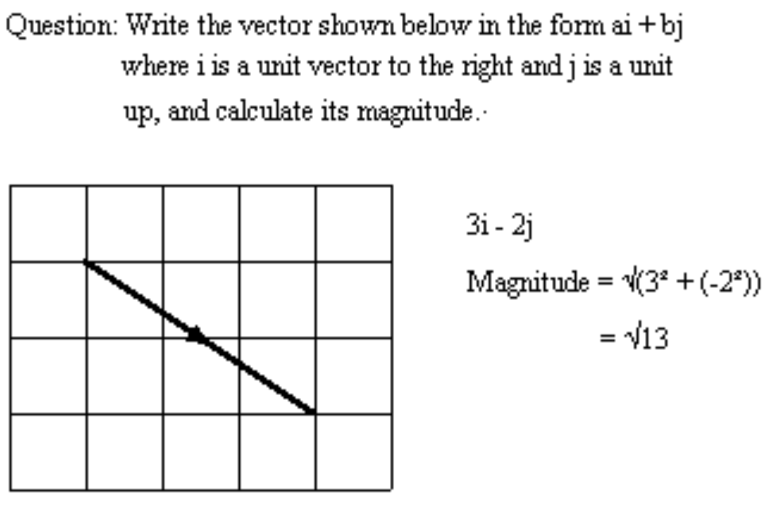


Figure 5. Question and answer involving the calculation of vector magnitude.

Absolute value objectified?

The class was working on textbook exercises involving the calculation of magnitude from vector components, and had been instructed to choose the Theorem of Pythagoras or ABS on their graphics calculators for solving the questions, and to do at least some questions with each method. The syntax for the function ABS, which signifies absolute value or magnitude, had been described with a numeric example (see Figure 6) on the worksheets that students had completed early in the lesson.

ABS is absolute value, which means magnitude or size, and is under the -*x* key.

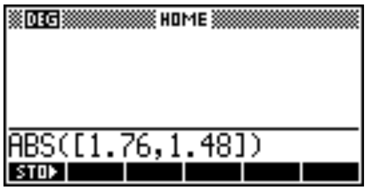


Figure 6. Extract from worksheet showing instructions for finding magnitude.

The action

Jenny and Katie were chatting as they worked when Jenny asked:

Jenny:	Katie, do you reckon you could put this into our G and T things, if you went 'ABS brackets **i j**'?

The 'G and T things' was an aplet or file that students had saved on their calculators with generalised Geometric and Trigonometric equations in it. Jenny's idea was to program her calculator to find magnitude so that, rather having to type in the whole ABS expression (see Figure 6) every time she came to a new question, she would have available the generalised form, ABS([I, J]), and only need to enter the values for I and J. She chose I and J to represent the **i** and **j** vector components, capitals being easier to access on the calculator keyboard than lower case letters.

Katie:	[still writing] Yes. That could work. [pause]
Jenny:	Ahh.
Katie:	Did it work?
Jenny:	I made a mistake.

Katie:	[leaning over to look at the screen of Jenny's calculator] How would you put in magnitude though?
Jenny:	Well I [pause], but it won't tell you. So okay, then you go like this [selecting = to make an equation].
Katie:	You have to make an M.
Jenny:	Yes. That would be okay. M could work. Then I could go ' = M'. [now she had ABS([I, J]) = M, which was the form required for the calculator to generate the magnitude after the I and J values were entered, see Figure 7]

Figure 7. The symbolic and numeric screens for the ABS aplet.

Cognitive perspective

Let us consider, up to this point in the lesson, how Katie and Jenny might have been thinking of magnitude and the function ABS which evaluated it as they worked through each question in the textbook (see Figure 5 for sample question with solution). First, their workbooks show they wrote the vector in question in component form, using the **i j** syntax. Then they wrote its magnitude as a decimal, without showing any working. Their methods are indicated by their ongoing conversation as they worked:

Jenny:	Right now we have to do some with Pythagoras. . . .
Katie:	I'm going to do the last four with Pythagoras.
Jenny:	Yes, Pythagoras doesn't take too long. But this [using the aplet] is really easy.

So, they obtained the magnitude answers in two different ways, as instructed. For some questions they used the Pythagoras Theorem, evaluating the square root expression (see Figure 5) on their calculator; so would have been seeing magnitude as the result of a calculation. The size of the square root answer (magnitude as a process) was the magnitude of the vector on the diagram for the question (magnitude as an object). Otherwise, they entered the function ABS into their calculator. For example, ABS([3, -2]) would yield the answer for the question in Figure 5. Here, were they thinking of ABS([3, -2]) in an operational or structural way? Had they objectified the function so that they they could think of its value as depending on 3 and -2 without needing to link the 3 and -2 to the squaring process? And what effect did generalising the ABS function have on their understanding? Sfard (1991), referring to Kaput (1979) and others, describes how the equality sign, ' = ', "can be regarded as a symbol of identity [of objects], or as a 'command' for executing operations" (p. 6). So, both structural and operational views of ABS were possible with the ABS([I,J])=M equation, just as they were for the expression ABS([3, -2]). Which view each student had at this stage is open to conjecture but, according to Sfard's (1991) theory, until students objectified the ABS function they wouldn't be able to operate on it, which is why I will speculate about their thinking a little further.

A few minutes later.

1.	Katie:	Do we have to take into account the negative [when using ABS]?
2.	Jenny:	Yes, methinks. Yes. Oh, no. You wouldn't, would you.
3. 4. 5. 6.	Katie:	You wouldn't, you just turn the table around there [turning the book 90 degrees anticlockwise so that the components in the diagram were in the positive **i** and **j**directions, see Figure 8]. It shouldn't matter. Its just like Pythagoras.
7.	Jenny:	This isn't Pythagoras. But it is the same thing, isn't it? Yes, it's okay.

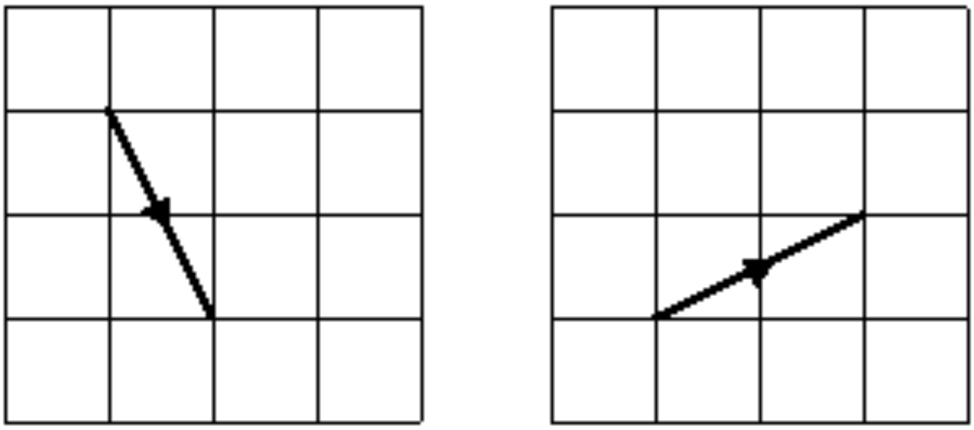


Figure 8. Rotating the vector diagram to make 1**i** - 2**j** appear like 2**i** +1**j**

In referring to the diagram (line 3), Katie might have been thinking of ABS on the calculator as being synonymous with the magnitude of the vector in the diagram, that is ABS as an object, but then she talked of it as the result of the Pythagoras process (line 5-6). Jenny confirmed Katie's decision by referring to the process (line 7).

8.	Me:	So, you are using these like second nature already.
9. 10. 11.	Jenny:	Not quite. Oh, Mrs Forster, when you have a negative one, do you have to put the negative in when you are working out with the ABS your calculator?
12. 13.	Me:	It doesn't matter, because a negative, what the calculator really does is Pythagoras.
14.	Jenny:	Pythagoras.
15.	Me:	And when you square it, the negative goes.
16.	Jenny:	The negative goes, yes. Right.

I moved away.

17.	Jenny:	[to herself]
18.		I might try that with the negative as well, to see if I get the same answer.
19.		[She reworked a problem on her calculator, using ABS([I,J])=M, assigning a negative sign to a positive component].
20.		Yes, it's the same.
21.		

While seeking further confirmation from me (line 9), where I referred to ABS in an operational (process) sense (line 13, 15), Jenny was perhaps moving towards a structural understanding of the ABS function: that is, thinking of it as a function of components, without referring back to the squaring process. Otherwise, why did she check the negative on her calculator (line 18) when, judging by what she had said to Katie (line 7) and me (line 16), she already knew that the process didn't need the negatives? But here we meet the problematic: there isn't a direct link between what is said in conversation and what is understood (Cobb, Gravemeijer, Yackel, McClain & Whitenack, 1997). However, Sfard (1991) discusses typical types of activity for each developmental level, and these provide a means to decipher the nature of Jenny and Katie's understanding. The act of generalising the ABS function from the specific was significant: "When function is concerned, the more capable the person becomes of playing with mapping as a whole, without looking into its specific values, the more advanced in the process of condensation he or she should be regarded" (Sfard, 1991, p. 19). Therefore, in theory, it seems that writing the generalisation ABS([I,J]) was an indication of structural thinking. In later lessons, Katie and Jenny demonstrated their structural understanding by operating on the function in insightful ways, which are described in the next section of the paper. The point of my discussion here is that, in this instance, students' movement from operational to structural understanding was difficult to discern. The movement perhaps is better described, like Sfard (1991) suggests, as a slow emergence of structural understanding from operational understanding.

Social perspective: Student-student interaction

Having focussed on Jenny and Katie's thinking, let us consider how the development of their understanding was related to their modes of interaction with each other. First, what might reflective discourse (Cobb, Boufi, McClain & Whitenack, 1997) mean in reference to the above episode? Returning to Katie's "Do we have to take account of the negative?" (line 1), she seemed by her question to tentatively understand that taking into account the negative signs was not necessary. She sought verification in conversation with Jenny. The *knowing-in-participation* (Sfard, 1998) she *shared* with Jenny, might have, on reflection, reduced the uncertainty in her understanding, but cannot be assumed to have done so (Cobb, et al., 1997). As well, the s*hared* knowing could have been perceived differently by each student (Cobb et al., 1997; Steffe, 1995). Did Katie sense Jenny's (line 2) "Yes, methinks. Yes. Oh, no. You wouldn't, would you" meant "no", for sure? For Jenny, how far was the balance in favor of "no" by the time she finished her sentence? When Katie checked out the hypothesis about the negatives by turning her book (line 3), were both girls convinced about the property?

When we consider Jenny, the issue of the negatives came to light in-participation (line 2), and she needed to test (line 9) the knowing she shared with Katie and her subsequent tentative understanding against the opinion of an *expert* (Lave & Wenger, 1991), where an expert is a person of wider experience. Then, she further checked her understanding with an example on her graphics calculator (line 18). However, like Cobb, Boufi, McClain and

Whitenack (1997) suggest for interpersonal interaction, there can be no guarantee of a direct link between interaction with technology and the psychological process of cognition. Mathematical understanding cannot be assumed to derive from or be evidenced by a student carrying out a procedure on a graphics calculator.

In summary, the episode so far illustrates that reflective discourse, in this instance centring on the use of technology, involves perpetual movement between the cognitive and social domains, with only the social being apparent to the outsider. An explanation for the apparent difference in uncertainty for the two students is that, for Katie, the conversation started with her operating in the cognitive domain and asking a question, while Jenny's involvement started in-participation with Katie. Katie may have already partly decided that taking account of the negatives wasn't necessary. Asking a friend a question perhaps has different implications for learning than answering a friend's question, with the type of question asked also being a determining factor.

Social perspective: Collective reflection

The teacher asked Jenny at the end of lesson to tell the class about the aplet. There had been time for other students to think of the approach but no-one had, even though all students were familiar with storing and using generalised trigonometric equations in their calculators.

Mr C:	. . . Right, now we have stopped, Jenny has volunteered to tell us something that she has discovered. So come up Jenny.
Jenny:	. . . and I put in a little aplet . . . so the aplet I got was, I put in ABS brackets, then I and J, and I just said M for magnitude. . . .
Mr C:	Okay. Would everyone like to try it? Does everyone think it is a good idea?
Leonie:	I think its brilliant.
Jenny:	It's not always a lot quicker than Home [where numerical calculations are routinely done], only if you have got lots and lots to do.

Jenny's explaining, which could be called didactic peer teaching, and other students' listening and subsequent move to use the generalisation perhaps doesn't exemplify the reflective discourse characterised by *collective reflection* that Cobb, Boufi, McClain and Whitenack (1997) describe. Collective reflection proceeds from students taking turns in conversation with each other, and with the teacher, to carry forward a question or suggestion, then they individually objectify their prior activity through reflection: the reflection is "supported and enabled by participation in the discourse" (Cobb et al., 1997, p. 264). Conversation necessarily involves one speaking and another or others listening (Schweickart, 1996) but perhaps it is the speaking that is most empowering.

Cobb (1998) discusses how, in a teaching experiment with Grade 7 students, a teacher capitalised on a student contribution, which represented an advancement in understanding, by interpreting other students' solutions in terms of it and asking yet others if they agreed. "As a consequence of this revoicing, it [the advancement] gradually became taken-as-shared" (p. 40). How widely understanding is 'shared' might depend on how many students get to take turns in conversation, or are ready to take a turn. So, assuming participation is an important facet of learning, teachers must ensure that all students "have a way to participate in the mathematical practices of the classroom community" (Cobb, 1998, p. 44). Small-group work, including working regularly with a partner over time so that rapport develops, allows more students to speak than does whole-class discussion. While contributions from those students who are at the forefront of thinking among the class might be of most benefit to the collective, it seems equally important for other individuals to have opportunities, and the capabilities and confidence, to voice their thinking. "One of the most important teaching goals is, therefore, the development of communicatively competent students" (Taylor, 1996, p. 167).

In summary, inferences drawn in the analysis are:

- evidence of shifts from operational to structural thinking can be difficult to discern from conversation,
- understanding that is 'shared' in conversation might be interpreted differently by each participant,
- different modes of discourse, including asking and answering questions, one student explaining to the class and others listening, students talking about their work in small groups as compared to whole-class discussion, might have differing potentials in facilitating the reflective processes in individual students,
- there is no guarantee of a link between interaction with technology and development of mathematical understanding.

Subsequent events

Nearly four weeks had passed since the first lesson on vectors. Problems involving displacement vectors, as well as velocity and force vectors of different directions and magnitude had been talked about and practised. Vectors and their attributes, vector components, magnitude and direction, had become taken-as-given among (all?) the class, providing a means for describing movement in two-dimensional space that was not open to question. Students frequently used the ABS function in an operational sense; as a command to carry out the Pythagorean process to evaluate absolute value or magnitude. But so far, during my classroom involvement I hadn't noticed any student making an "instantaneous quantum leap" (Sfard, 1991, p. 20) to operate *on* the function, which would reflect reification of the function to think of it as an entity in its own right.

To check that I hadn't missed a shift in the discourse that reflected reification, I manually searched the transcripts of whole-class discussion and of the conversations of the five students whose individual participation I had recorded. Then I did an electronic search on 'aplet', 'ABS' and 'absolute' in the transcripts. The searches showed that the ABS function was referred to on six different occasions in the four-week period. Five times the reference was in the context of evaluating magnitude from vector components and once was when Jenny argued during class discussion that negatives didn't need to be taken into account when calculating magnitude; that is, students were holding operational views of the ABS function. Then a response in whole-class discussion caught my attention.

A shift in class discussion

I was observing the class and they were working on algebraic questions devoid of context. A textbook question asked for a vector, magnitude 25, parallel to 3**i** - 4**j**. Jenny solved the problem by operating *on* I, J and M, keeping them in proportion. She explained her method to the class:

Jenny:	What I did was go to my little aplet that works out magnitude, and if you increase

	or decrease the **i** and **j** components in proportion, so 3 and -4 will be 6 and -8 and 9 and -12. (ABS([3, -4])=5, ABS([6, -8])=10, ABS([9, -12])=15) . . . . That's basically what I did, I put in 3 and -4 and it came out as 5, so I said okay well if it going to be five times that, so I said 15 and -20, and it worked. [see Figure 9 for visual reasoning that is consistent with her method]

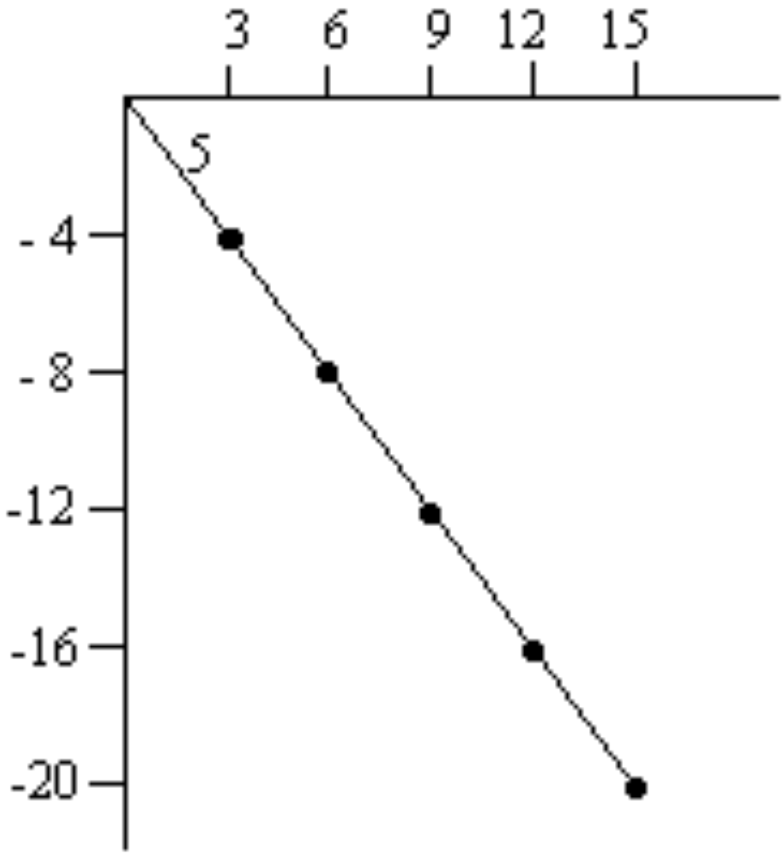


Figure 9. Magnitude and components of a vector, magnitude 5, operated on in proportion.

Jenny's solution illustrates reification of the ABS function: she had moved to the third stage of conceptualising the function. In the first lesson, after encountering the ABS on the worksheet, she had advanced through the interiorisation stage to the condensation level. Then, four weeks later, we see her 'jumping' to the reification stage, stimulated to do so by the question that was asked.

The purpose in writing this paper of explaining students' advancement in understanding while co-opting technology for problem solving seems to have been achieved to the extent of mapping one cycle of development for a single concept. Progress in technology-assisted learning has been matched to Sfard's (1991) three developmental stages, with shifts in discourse providing evidence of progress: reflective discourse (Cobb, Boufi, McClain and Whitenack (1997), with its shifts in the way a concept is referred to, seems to evidence the development of mathematical understanding. However, in its emphasis on conversation, the analysis begs several questions. 'Did students evidence structural understanding of the ABS function other than in conversation?'; 'Did some students learn effectively without high degrees of involvement in conversation?'; and 'Can any events in the four weeks from the first lesson when Katie and Jenny designed the aplet explain Jenny's 'jump' to the reify the ABS function?'. These questions are now considered briefly.

Further exploration

The only other instance that I have identified so far of a student having reached the reification stage is in the students' assessment test on the two-dimensional vector topic. One test question asked: If **b** = 3**i** - 2**j** and **c** = w**i** -3**j**, find the value of w if **c** is parallel to **b**. The question type had not been encountered in class so could be considered as non-routine. When perusing the students' test scripts I noticed that Katie had no working other than a diagram to support her answer (see Figure 10).

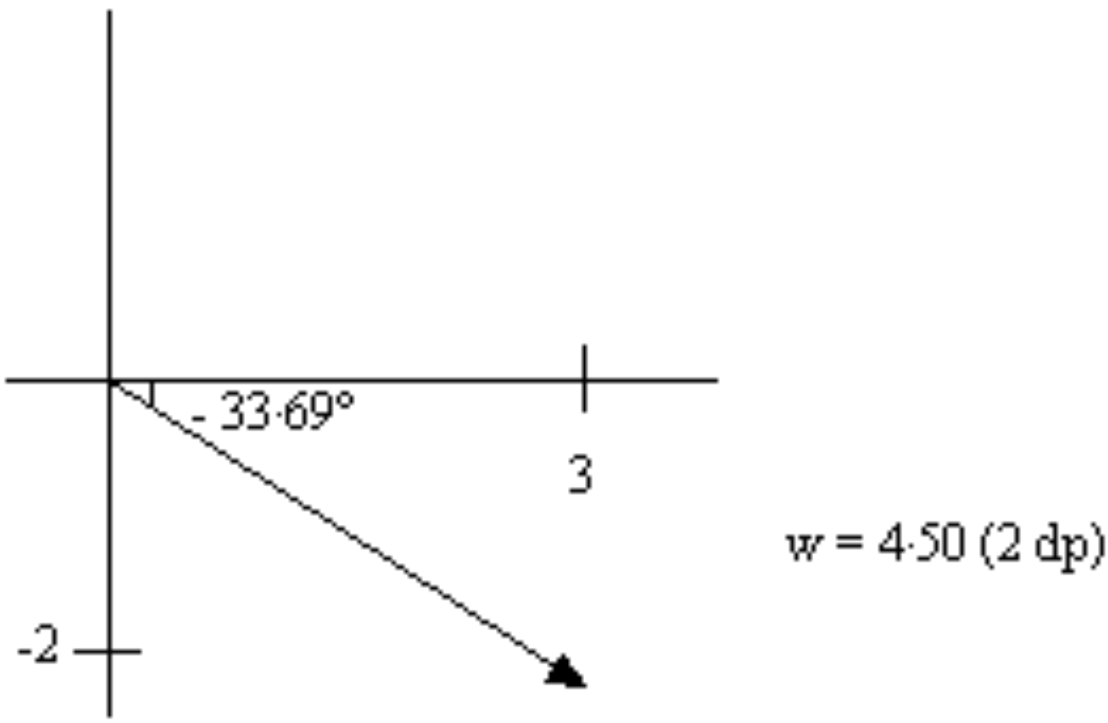


Figure 10. Katie's working for the test question.

After the test I asked Katie to explain her method to me. In her calculator, as well as the ABS equation for finding magnitude, she had an equation ARG((I, J)) = A for finding angles. She had entered I = 3, and J = -2 in it, solving the equation to show ARG((3, -2)) = -33.69° . Then she changed J to -3, leaving A = -33.69° , and solved for I to show ARG((4.50, -3)) = -33.69° . She had operated *on* A to obtain I, which parallels operating on the absolute value M to find I. Both Katie and Jenny had jumped to the reification stage of understanding

(Sfard, 1991) to operate on the ABS function or the similar ARG function as an object, Katie under the pressure of a test. That they reached this level before others did (later in the course) can be explained by them having originated the idea of generalising the vector calculations, while in conversation.

Although Katie and Jenny were by far the most communicative among the eighteen students, others in the class achieved similar marks in the test to them, an indication of comparable mathematical understanding (or perhaps reflecting the inadequacy of summative assessment). It might be that students can individually make strong progress by reflectively listening and writing, without high levels of verbalising their thoughts. And, as Sfard comments (Sfard, Nesher, Streefland, Cobb & Mason, 1998), "the loneliness of professional mathematicians is notorious" (p. 42). This isn't to say that students who work in comparative isolation wouldn't have their learning enhanced with conversation, that is, with speaking rather than predominantly listening. In addition, when a student keeps her thoughts to herself, the person they sit next to and the group as a whole don't benefit from her insight, so progress (including the student's own progress) might be less or slower than otherwise possible.

Returning to the test question discussed above, six other students, besides Katie and including Jenny, solved the question correctly. They used proportion. For example, Jenny wrote: -2 : -3, 3 : w, \ w = 4.5. The question couldn't be solved using the ABS aplet, but maybe Jenny's previous mention of using proportion (see description above) with the ABS function influenced the students' choice of method: proportion was not named in class for the vector topic at any other time that I can remember and doesn't show up in the video and audio transcripts. Similarly, connection of ideas can explain Jenny and Katie's 'jump' to the reification level of understanding. During the four weeks that elapsed between the students first holding a process view of the ABS function and them later operating on the function, or the similar ARG function, they had solved problems that involved operating on magnitude as an object: for example multiplying speed (magnitude of a velocity vector) by time to obtain distance. Even in the first lesson, students potentially moved between operational and structural views of magnitude (see earlier discussion). Connecting work on magnitude to the ABS function perhaps enabled its reification, so reification mightn't always involve the quantum leap that Sfard (1991) suggests. Rather, movement from the structural to reification stage might, as for the movement from operational to structural thinking, be better described as an emergence. The emergence could be cast, in terms of Ausubelian (1968) theory, as involving a progressive linkage of concepts in "the whole matrix of interconnected concepts" (Novak, 1978, p. 6). The reification stage itself would then involve further linkages to allow increasingly sophisticated operations on the objectified concept.

In summary, implications for teaching and research that can be drawn from the analysis in this section are:

- for students to advance from one developmental stage to the next, appropriate questions need to be provided. When learning is technology-assisted, traditional questions might be solved with new methods, or new questions might be devised to encompass new approaches,
- movement between developmental stages can be evidenced and constituted by writing as well as by conversation,
- engaging in conversation rather than working in isolation potentially benefits others,

- conceptual development is not necessarily a linear process.

Conclusion

The theoretical construct of reflective discourse (Cobb, Boufi, McClain & Whitenack, 1997), with shifts in discourse consistent with Sfard's (1991) theory of mathematical development, was a valuable tool of analysis in allowing an explanation of students' technology-based learning. However, identifying shifts in students' conversations that indicated a move from operational to structural understanding was difficult and, to acknowledge the complexity of learning, the analysis needs to be supplemented with a description of the long processes of students connecting new concepts to existing knowledge during the condensation and reification stages.

However, the classroom episode involving the design of a graphics calculator aplet, with the two students Katie and Jenny at the centre of the action, demonstrated what reflective discourse might achieve. With the specified numeric form of the absolute function ABS([1.76, 1.48]) acting as a coherence object (Roth & Tobin, in press), the students together generalised the function. The design of the aplet and subsequent discussion of the Pythagorean process behind the ABS function, and about whether or not to take account of the negatives, involved them taking turns in speaking and individually reflecting on the meaning of the conversation. Through engaging in both social and cognitive processes they pushed their thinking forward. Other students in the class learnt about the aplet from a transmissionist type telling and used the aplet in problem solving, but my observations suggest that Jenny and Katie were again the first to 'jump' forward to a more sophisticated level of thinking, to operate on the function (or the similar ARG function) as an object. The advances of the two students can be cast as having been constituted by their active and reflective engagement in discussion, which was their typical way of working.

Viewing learning in terms of social interaction indicated that 'shared' meaning in conversation might be interpreted differently by each participant and that some modes of interaction, for example asking as compared to answering questions, might impact differently on an individual's learning. The inquiry also highlighted that written as well as verbal action evidences and constitutes students' mathematical development; and the link between cognition and interaction with technology is indirect, like for interpersonal interaction. Implications from this inquiry are: (a) for research, explaining the learning process requires acknowledging the inter-relationship between students' cognition, social interaction and use of technology; and (b) for teaching, the use of technology might be enhanced by conversation, not only for the individual but for the group, and that curriculum materials need to provide students with the opportunity to move forward in their thinking and to utilise the higher-level capabilities of the technologies. On the assumption that reflective discourse is potentially empowering, teachers should avoid pre-empting the way graphics calculators can be used to solve problems and instead encourage students to discuss with each other how they are using the technology.

In view of the increasing accommodation of technology into school curriculae, longer-term investigation of larger samples of students seems warranted in order to explore the interdependence of the cognitive and social aspects of learning, in the presence of technology. Inquiries in context of upper-secondary mathematics would be particularly appropriate as it is an area that seems to receive less attention in the literature than elementary mathematics.

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