



Murdoch
UNIVERSITY

MURDOCH RESEARCH REPOSITORY

<http://researchrepository.murdoch.edu.au>

This is the author's final version of the work, as accepted for publication following peer review but without the publisher's layout or pagination.

Bucci, M., Deluca, A., Glen, A. and Zamboni, L. (2009) A connection between palindromic and factor complexity using return words. *Advances in Applied Mathematics*, 42 (1). pp. 60-74.

<http://researchrepository.murdoch.edu.au/3806>

Copyright © 2008 Elsevier Inc.
It is posted here for your personal use. No further distribution is permitted.

A connection between palindromic and factor complexity using return words^{**}

Michelangelo Bucci* Alessandro De Luca* Amy Glen† Luca Q. Zamboni‡

June 8, 2008

Abstract

In this paper we prove that for any infinite word \mathbf{w} whose set of factors is closed under reversal, the following conditions are equivalent:

(I) all complete returns to palindromes are palindromes;

(II) $\mathcal{P}(n) + \mathcal{P}(n + 1) = \mathcal{C}(n + 1) - \mathcal{C}(n) + 2$ for all n ,

where \mathcal{P} (resp. \mathcal{C}) denotes the *palindromic complexity* (resp. *factor complexity*) function of \mathbf{w} , which counts the number of distinct palindromic factors (resp. factors) of each length in \mathbf{w} .

Keywords: return word; palindrome; palindromic complexity; factor complexity; Rauzy graph; rich word.

MSC (2000): 68R15.

1 Introduction

Given an infinite word \mathbf{w} , let $\mathcal{P}(n)$ (resp. $\mathcal{C}(n)$) denote the *palindromic complexity* (resp. *factor complexity*) of \mathbf{w} , i.e., the number of distinct palindromic factors (resp. factors) of \mathbf{w} of length n . In [1], J.-P. Allouche, M. Baake, J. Cassaigne, and D. Damanik established the following inequality relating the palindromic and factor complexities of a non-ultimately periodic infinite word:

$$\mathcal{P}(n) \leq \frac{16}{n} \mathcal{C}\left(n + \left\lfloor \frac{n}{4} \right\rfloor\right) \quad \text{for all } n \in \mathbb{N}.$$

More recently, using *Rauzy graphs*, P. Baláži, Z. Masáková, and E. Pelantová [5] proved that for any uniformly recurrent infinite word whose set of factors is closed under reversal,

$$\mathcal{P}(n) + \mathcal{P}(n + 1) \leq \mathcal{C}(n + 1) - \mathcal{C}(n) + 2 \quad \text{for all } n \in \mathbb{N}. \quad (1.1)$$

They also provided several examples of infinite words for which $\mathcal{P}(n) + \mathcal{P}(n + 1)$ always reaches the upper bound given in relation (1.1). Such infinite words include *Arnoux-Rauzy sequences*,

*Dipartimento di Matematica e Applicazioni “R. Caccioppoli”, Università degli Studi di Napoli Federico II, Via Cintia, Monte S. Angelo, I-80126, Napoli, ITALY (micbucci, alessandro.deluca@unina.it).

†**Corresponding author:** LaCIM, Université du Québec à Montréal, C.P. 8888, succursale Centre-ville, Montréal, Québec, H3C 3P8, CANADA (amy.glen@gmail.com). Supported by CRM, ISM, and LaCIM.

‡Université de Lyon, Université Lyon 1, CNRS UMR 5208 Institut Camille Jordan, Bâtiment du Doyen Jean Braconnier, 43, blvd du 11 novembre 1918, F-69622 Villeurbanne Cedex, FRANCE and Reykjavik University, School of Computer Science, Kringlan 1, 103 Reykjavik, ICELAND (luca.zamboni@wanadoo.fr).

**To appear in *Advances in Applied Mathematics*. Submitted: February 9, 2008. Accepted: March 25, 2008.

complementation-symmetric sequences, certain words associated with β -expansions where β is a *simple Parry number*, and a class of words coding r -interval exchange transformations.

In this paper we give a characterization of all infinite words with factors closed under reversal for which the equality $\mathcal{P}(n) + \mathcal{P}(n + 1) = \mathcal{C}(n + 1) - \mathcal{C}(n) + 2$ holds for all n : these are exactly the infinite words with the property that all ‘complete returns’ to palindromes are palindromes. Given a finite or infinite word w and a factor u of w , we say that a factor r of w is a *complete return* to u in w if r contains exactly two occurrences of u , one as a prefix and one as a suffix. Return words play an important role in the study of minimal subshifts; see [12, 13, 14, 15, 20, 24].

Our main theorem is the following:

Theorem 1.1. *For any infinite word \mathbf{w} whose set of factors is closed under reversal, the following conditions are equivalent:*

- (I) *all complete returns to any palindromic factor of \mathbf{w} are palindromes;*
- (II) $\mathcal{P}(n) + \mathcal{P}(n + 1) = \mathcal{C}(n + 1) - \mathcal{C}(n) + 2$ for all $n \in \mathbb{N}$.

Recently, in [19], it was shown that property (I) is equivalent to every factor u of \mathbf{w} having exactly $|u| + 1$ distinct palindromic factors (including the empty word). Such words are ‘rich’ in palindromes in the sense that they contain the maximum number of different palindromic factors. Indeed, X. Droubay, J. Justin, and G. Pirillo [10] observed that any finite word w of length $|w|$ contains at most $|w| + 1$ distinct palindromes.

The family of finite and infinite words having property (I) are called *rich words* in [19]. In independent work, P. Ambrož, C. Frougny, Z. Masáková, and E. Pelantová [2] have considered the same class of words which they call *full words*, following earlier work of S. Brlek, S. Hamel, M. Nivat, and C. Reutenauer in [6].

Rich words encompass the well-known family of *episturmian words* originally introduced by X. Droubay, J. Justin, and G. Pirillo in [10] (see Section 4 for more details). Another special class of rich words consists of S. Fischler’s sequences with ‘abundant palindromic prefixes’, which were introduced and studied in [16] in relation to Diophantine approximation (see also [17]). Other examples of rich words that are neither episturmian nor of ‘Fischler type’ include: non-recurrent rich words, like $abbbb \cdots$ and $abaabaaabaaaab \cdots$; the periodic rich infinite words: $(aab^k aabab)(aab^k aabab) \cdots$, with $k \geq 0$; the non-ultimately periodic recurrent rich infinite word $\psi(\mathbf{f})$ where $\mathbf{f} = abaababaaba \cdots$ is the *Fibonacci word* and ψ is the morphism: $a \mapsto aab^k aabab$, $b \mapsto bab$; and the recurrent, but not uniformly recurrent, rich infinite word generated by the morphism: $a \mapsto aba$, $b \mapsto bb$. (See [19] for these examples and more.)

From the work in [10, 19], we have the following equivalences.

Proposition 1.2. *A finite or infinite word w is rich if equivalently:*

- *all complete returns to any palindromic factor of w are palindromes;*
- *every factor u of w contains exactly $|u| + 1$ distinct palindromes;*
- *the longest palindromic suffix of any prefix p of w occurs exactly once in p .*

From the perspective of richness, our main theorem can be viewed as a characterization of *recurrent rich infinite words* since any rich infinite word is recurrent if and only if its set of factors is closed under reversal (see [19] or Remark 2.1). Interestingly, the proof of Theorem 1.1 relies upon another new characterization of rich words (Proposition 2.3), which is useful for establishing the key step, namely that the so-called *super-reduced Rauzy graph* is a tree. This answers a claim

made in the last few lines of [5] where it was remarked that the Rauzy graphs of words satisfying equality (II) must have a very special form.

After some preliminary definitions and results in the next section, Section 3 is devoted to the proof of Theorem 1.1 and some interesting consequences are proved in Section 4.

2 Preliminaries

2.1 Notation and terminology

In this paper, all words are taken over a finite *alphabet* \mathcal{A} , i.e., a finite non-empty set of symbols called *letters*. A *finite word* over \mathcal{A} is a finite sequence of letters from \mathcal{A} . A (right) *infinite word* \mathbf{x} is a sequence indexed by \mathbb{N}_+ with values in \mathcal{A} , i.e., $\mathbf{x} = x_1x_2x_3 \cdots$ with each $x_i \in \mathcal{A}$. For easier reading, infinite words are hereafter typed in boldface to distinguish them from finite words.

Given a finite word $w = x_1x_2 \cdots x_m$ (where each x_i is a letter), the *length* of w , denoted by $|w|$, is equal to m . By convention, the *empty word* ε is the unique word of length 0. We denote by \tilde{w} the *reversal* of w , given by $\tilde{w} = x_m \cdots x_2x_1$. If $w = \tilde{w}$, then w is called a *palindrome*.

A finite word z is a *factor* of a finite or infinite word w if $w = uzv$ for some words u, v . In the special case $u = \varepsilon$ (resp. $v = \varepsilon$), we call z a *prefix* (resp. *suffix*) of w . If $u \neq \varepsilon$ and $v \neq \varepsilon$, then we say that z is an *interior factor* of $w = uzv$. Moreover, z is said to be a *central factor* of w if $|u| = |v|$. We say that z is *unioccurrent* in w if z occurs exactly once in w . For any finite or infinite word w , the set of all factors of w is denoted by $F(w)$ and we denote by $F_n(w)$ the set of all factors of w of length n , i.e., $F_n(w) := F(w) \cap \mathcal{A}^n$ (where $|w| \geq n$ if w is finite). We say that $F(w)$ is *closed under reversal* if for any $u \in F(w)$, $\tilde{u} \in F(w)$.

A factor of an infinite word \mathbf{w} is *recurrent* in \mathbf{w} if it occurs infinitely often in \mathbf{w} , and \mathbf{w} itself is said to be *recurrent* if all of its factors are recurrent in it. Furthermore, \mathbf{w} is *uniformly recurrent* if any factor of \mathbf{w} occurs infinitely many times in \mathbf{w} with bounded gaps.

Remark 2.1. A noteworthy fact (proved in [19]) is that a rich infinite word is recurrent if and only if its set of factors is closed under reversal.

More generally, we have the following well-known result:

Proposition 2.2 (folklore). *If \mathbf{w} is an infinite word with $F(\mathbf{w})$ closed under reversal, then \mathbf{w} is recurrent.*

Proof. Consider some occurrence of a factor u in \mathbf{w} and let v be a prefix of \mathbf{w} containing u . As $F(\mathbf{w})$ is closed under reversal, $\tilde{v} \in F(\mathbf{w})$. Thus, if v is long enough, there is an occurrence of \tilde{u} strictly on the right of this particular occurrence of u in \mathbf{w} . Similarly u occurs on the right of this \tilde{u} and thus u is recurrent in \mathbf{w} . \square

2.2 Key results

We now prove two useful results, the first being a new characterization of rich words.

Proposition 2.3. *A finite or infinite word w is rich if and only if, for each factor $v \in F(w)$, any factor of w beginning with v and ending with \tilde{v} and not containing v or \tilde{v} as an interior factor is a palindrome.*

Proof. ONLY IF: Consider any factor $v \in F(w)$ and let u be a factor of w beginning with v and ending with \tilde{v} and not containing v or \tilde{v} as an interior factor. If v is a palindrome, then either

$u = v = \tilde{v}$ (in which case u is clearly a palindrome), or u is a complete return to v in w , and hence u is (again) a palindrome by Proposition 1.2. Now assume that v is not a palindrome.

Suppose by way of contradiction that u is not a palindrome and let p be the longest palindromic suffix of u (which is unioccurrent in u by richness). Then $|p| < |u|$ as u is not a palindrome. If $|p| > |v|$, then \tilde{v} is a proper suffix of p , and hence v is a proper prefix of p . But then v is an interior factor of u , a contradiction. On the other hand, if $|p| \leq |v|$, then $|p| \neq |v|$ and p is a proper suffix of \tilde{v} (as \tilde{v} is not a palindrome), and hence p is a proper prefix of v . Thus p is both a prefix and a suffix of u ; in particular p is not unioccurrent in u , a contradiction.

IF: The given conditions tell us that any complete return to a palindromic factor $v (= \tilde{v})$ of w is a palindrome. Hence w is rich by Proposition 1.2. \square

Proposition 2.4. *Suppose w is a rich word. Then, for any non-palindromic factor v of w , \tilde{v} is a unioccurrent factor of any complete return to v in w .*

Proof. Let r be a complete return to v in w and let p be the longest palindromic suffix of r . Then $|p| > |v|$; otherwise, if $|p| \leq |v|$, then p would occur at least twice in r (as a suffix of each of the two occurrences of v in r), which is impossible as r is rich. Thus v is a proper suffix of p , and hence \tilde{v} is a proper prefix of p . So \tilde{v} is clearly an interior factor of r .

It remains to show that \tilde{v} is unioccurrent in r . Arguing by contradiction, we suppose that \tilde{v} occurs more than once in r . Then a complete return r' to \tilde{v} occurs as a proper factor of r . Using the same reasoning as above, v is an interior factor of r' , and hence an interior factor of r , contradicting the fact that r is a complete return to v . Thus \tilde{v} is unioccurrent in r . \square

Note. The above proposition tells us that for any factor v of a rich word w , occurrences of v and \tilde{v} alternate in w .

3 Proof of Theorem 1.1

Following the method of Baláži *et al.* [5], a key tool for the proof of our main theorem is the notion of a *Rauzy graph*, defined as follows. Given an infinite word \mathbf{w} , the *Rauzy graph of order n* for \mathbf{w} , denoted by $\Gamma_n(\mathbf{w})$, is the directed graph with set of vertices $F_n(\mathbf{w})$ and set of edges $F_{n+1}(\mathbf{w})$ such that an edge $e \in F_{n+1}(\mathbf{w})$ starts at vertex v and ends at a vertex v' if and only if v is a prefix of e and v' is a suffix of e . For a vertex v , the *out-degree* of v (denoted by $\deg^+(v)$) is the number of distinct edges leaving v , and the *in-degree* of v (denoted by $\deg^-(v)$) is the number of distinct edges entering v . More precisely:

$$\deg^+(v) = \#\{x \in \mathcal{A} \mid vx \in F_{n+1}(\mathbf{w})\} \quad \text{and} \quad \deg^-(v) = \#\{x \in \mathcal{A} \mid xv \in F_{n+1}(\mathbf{w})\}.$$

We observe that, for all $n \in \mathbb{N}$,

$$\sum_{v \in F_n(\mathbf{w})} \deg^+(v) = \#F_{n+1}(\mathbf{w}) = \sum_{v \in F_n(\mathbf{w})} \deg^-(v).$$

(Note that $\#F_{n+1}(\mathbf{w}) = \mathcal{C}(n+1)$.) Hence

$$\mathcal{C}(n+1) - \mathcal{C}(n) = \sum_{v \in F_n(\mathbf{w})} (\deg^+(v) - 1) = \sum_{v \in F_n(\mathbf{w})} (\deg^-(v) - 1). \quad (3.1)$$

It is therefore easy to see that a factor $v \in F_n(\mathbf{w})$ positively contributes to $\mathcal{C}(n+1) - \mathcal{C}(n)$ if and only if $\deg^+(v) \geq 2$, i.e., if and only if there exist at least two distinct letters a, b such that va ,

$vb \in F_{n+1}(\mathbf{w})$, in which case v is said to be a *right-special* factor of \mathbf{w} . Similarly, a factor $v \in F_n(\mathbf{w})$ is said to be a *left-special* factor of \mathbf{w} if there exist at least two distinct letters a, b such that $av, bv \in F_{n+1}(\mathbf{w})$. A factor of \mathbf{w} is said to be *special* if it is either left-special or right-special (not necessarily both). With this terminology, if we let $S_n(\mathbf{w})$ denote the set of special factors of \mathbf{w} of length n , then formula (3.1) may be expressed as:

$$\mathcal{C}(n+1) - \mathcal{C}(n) = \sum_{v \in S_n(\mathbf{w})} (\deg^+(v) - 1) \quad \text{for all } n \in \mathbb{N}. \quad (3.2)$$

Using similar terminology to that in [5], a directed path P in the Rauzy graph $\Gamma_n(\mathbf{w})$ is said to be a *simple path of order n* if it begins with a special factor v and ends with a special factor v' and contains no other special factors, i.e., P is a directed path of the form vv' or $vz_1 \cdots z_k v'$ where each z_i is a non-special factor of length n . A special factor $v \in S_n(\mathbf{w})$ is called a *trivial* simple path of order n .

In what follows, we use the following terminology for paths. Hereafter, “path” should be taken to mean “directed path”.

Definition 3.1. Suppose \mathbf{w} is an infinite word and let $P = v \cdots v'$ be a path in $\Gamma_n(\mathbf{w})$.

- The first vertex v (resp. last vertex v') is called the *initial vertex* (resp. *terminal vertex*) of P .
- A vertex of P that is neither an initial vertex nor a terminal vertex of P is called an *interior vertex* of P .
- P is said to be a *non-trivial path* if it consists of at least two distinct vertices.
- The *reversal* \tilde{P} of the path P is the path obtained from P by reversing all edge labels (and arrows) and all labels of vertices.
- We say that P is *palindromic* (or that P is *invariant under reversal*) if $P = \tilde{P}$.

Note. Given a path P in $\Gamma_n(\mathbf{w})$, the reversal of P does not necessarily exist in $\Gamma_n(\mathbf{w})$.

Suppose $P = w_1 w_2 \cdots w_k$ is a non-trivial path in $\Gamma_n(\mathbf{w})$, and for each i with $1 \leq i \leq k$, let a_i and b_i denote the respective first and last letters of w_i . Then, by the definition of $\Gamma_n(\mathbf{w})$, we have $w_1 b_2 \cdots b_k = a_1 \cdots a_{k-1} w_k$. We call this word the *label* of the path P , denoted by ℓ_P . Note that the i -th shift of $\ell_P := w_1 b_2 \cdots b_k$ begins with w_{i+1} for all i with $1 \leq i \leq k-1$.

For our purposes, it is convenient to consider the *reduced Rauzy graph of order n* , denoted by $\Gamma'_n(\mathbf{w})$, which is the directed graph obtained from $\Gamma_n(\mathbf{w})$ by replacing each *simple path* $P = w_1 w_2 \cdots w_{k-1} w_k$ with a directed edge $w_1 \rightarrow w_k$ labelled by ℓ_P . Thus the set of vertices of $\Gamma'_n(\mathbf{w})$ is $S_n(\mathbf{w})$. For example, consider the (rich) *Fibonacci word*:

$$\mathbf{f} = abaababaabaababaababaabaababaabaababaabaababaaba \cdots$$

which is generated by the *Fibonacci morphism* $\varphi : a \mapsto ab, b \mapsto a$. The reduced Rauzy graph $\Gamma'_2(\mathbf{f})$ consists of the two (special) vertices: ab, ba and three directed edges: $ab \rightarrow ba, ba \rightarrow ab, ab \rightarrow ab$ with respective labels: $aba, baab, bab$.

Lemma 3.2. *Let \mathbf{w} be a rich infinite word and suppose $P = w_1 w_2 \cdots w_k$ is a non-trivial path in $\Gamma_n(\mathbf{w})$ with $k \geq 2$. Then the label $\ell_P = w_1 b_2 \cdots b_k$ is a rich word.*

Proof. We proceed by induction on the number of vertices k in P . The lemma is clearly true for $k = 2$ since $\ell_P = w_1b_2$ is a factor of \mathbf{w} of length $n + 1$. Now suppose $k \geq 3$ and assume that the label of any path consisting of $k - 1$ vertices is rich. Consider any path consisting of k vertices, namely $P = w_1w_2 \cdots w_k$, and suppose by way of contradiction that its label $\ell_P = w_1b_2 \cdots b_k$ is not rich. Then the longest palindromic prefix p of ℓ_P occurs more than once in ℓ_P . Hence there exists a complete return r to p which is a prefix of ℓ_P . It follows that $r = \ell_P$, otherwise r would be a factor of the prefix $u := w_1b_2 \cdots b_{k-1}$ of ℓ_P , and hence a palindrome since u is rich by the induction hypothesis. But this contradicts the maximality of the palindromic prefix p . So ℓ_P is a non-palindromic complete return to p . Let q be the longest palindromic prefix of u (which is unioccurrent in u by richness). If $|p| > |q|$, then q is a proper prefix of p , and hence q occurs more than twice in u , a contradiction. On the other hand, if $|p| \leq |q|$, then p is a prefix of q , and hence p is an interior factor of ℓ_P (occurring as a suffix of q), a contradiction. Thus ℓ_P is rich, as required. \square

The proof of Theorem 1.1 relies upon the following extensions of Propositions 2.3–2.4 to paths.

Lemma 3.3. (Analogue of Proposition 2.3.) *Suppose \mathbf{w} is a rich infinite word and let v be any factor of \mathbf{w} of length n . If $P = v \cdots \tilde{v}$ is a path from v to \tilde{v} in $\Gamma_n(\mathbf{w})$ that does not contain v or \tilde{v} as an interior vertex, then P is palindromic. This property also holds for paths in $\Gamma'_n(\mathbf{w})$.*

Proof. We first observe that if P consists of a single vertex, then $P = v = \tilde{v}$, and hence P is palindromic. Now suppose P is a non-trivial path. If $P = v\tilde{v}$, then P is clearly palindromic. So suppose $P = vz_1 \cdots z_k \tilde{v}$ where the z_i are factors of \mathbf{w} of length n . By definition, the label $\ell_P = vb_1 \cdots b_k b_{k+1}$ begins with v and ends with \tilde{v} and contains neither v nor \tilde{v} as an interior factor (otherwise P would contain v or \tilde{v} as an interior vertex, which is not possible). Thus, as ℓ_P is rich (by Lemma 3.2), it follows that ℓ_P is a palindrome by Proposition 2.3; whence P must be invariant under reversal too. It is easy to see that this property is also true for paths in the reduced Rauzy graph $\Gamma'_n(\mathbf{w})$. \square

Lemma 3.4. (Analogue of Proposition 2.4.) *Suppose \mathbf{w} is a rich infinite word and let v be any non-palindromic factor of \mathbf{w} of length n . If $P = v \cdots v$ is a non-trivial path in $\Gamma_n(\mathbf{w})$ that does not contain v as an interior vertex, then P passes through \tilde{v} exactly once. This property also holds for paths in $\Gamma'_n(\mathbf{w})$.*

Note. Of particular usefulness is the fact that any path from v to v must pass through \tilde{v} .

Proof. Let us write $P = vz_1 \cdots z_kv$ where the z_i are factors of \mathbf{w} of length n . By definition, the label $\ell_P = vb_1 \cdots b_k b_{k+1}$ contains exactly two occurrences of v , one as a prefix and one as a suffix (otherwise, if ℓ_P contained v as an interior factor, then v would be an interior vertex of P , which is not possible). Thus, as ℓ_P is rich (by Lemma 3.2), it follows that \tilde{v} is a unioccurrent (interior) factor of ℓ_P by Proposition 2.4; whence P passes through \tilde{v} exactly once. It is easy to see that this property is also true for paths in the reduced Rauzy graph $\Gamma'_n(\mathbf{w})$. \square

3.1 (I) implies (II)

Suppose \mathbf{w} is an infinite word with $F(\mathbf{w})$ closed under reversal and satisfying property (I). Then \mathbf{w} is recurrent by Proposition 2.2 (i.e., \mathbf{w} is a recurrent rich infinite word). Moreover, recurrence implies that for all n , the Rauzy graph $\Gamma_n(\mathbf{w})$ is strongly connected, i.e., there exists a directed path from any vertex v to every other vertex v' in $\Gamma_n(\mathbf{w})$.

Fix $n \in \mathbb{N}$ and let us now consider the *super-reduced Rauzy graph of order n* , denoted by $\Gamma_n''(\mathbf{w})$, whose set of vertices consists of all $[v] := \{v, \tilde{v}\}$ where v is any special factor of length n . Any two distinct vertices $[v], [w]$ (with $v \notin \{w, \tilde{w}\}$) are joined by an undirected edge with label $[\ell_P] := \{\ell_P, \ell_{\tilde{P}}\}$ if P or \tilde{P} is a simple path beginning with v or \tilde{v} and ending with w or \tilde{w} . For example, in the case of the Fibonacci word, $\Gamma_2''(\mathbf{f})$ consists of only one vertex: $[ab]$. In general, the super-reduced Rauzy graph consists of more than one vertex and may contain multiple edges between vertices.

Suppose $\Gamma_n''(\mathbf{w})$ consists of s vertices; namely $[v_i], i = 1, \dots, s$. Since $\Gamma_n(\mathbf{w})$ is strongly connected (by recurrence), $\Gamma_n''(\mathbf{w})$ is connected; thus it contains at least $s - 1$ edges.

Now, from Lemma 3.3, we know that if v is a special factor, any simple path from v to \tilde{v} is palindromic (i.e., invariant under reversal). Moreover, by closure under reversal, if there exists a simple path P from a special factor v to a special factor w , with $v \notin \{w, \tilde{w}\}$, then there is also a simple path from \tilde{w} to \tilde{v} (namely, the reversal of the path P). Neither of these simple paths is palindromic.

We thus deduce that there exist *at least* $2(s - 1)$ non-trivial simple paths in the Rauzy graph $\Gamma_n(\mathbf{w})$ that are non-palindromic (i.e., not invariant under reversal). In fact, we will show that there are exactly $2(s - 1)$ non-trivial simple paths of order n that are non-palindromic. Indeed, if this true then, as each palindromic factor of length n or $n + 1$ is a central factor of a (unique) palindromic simple path of order n , we have:

$$\mathcal{P}(n) + \mathcal{P}(n + 1) = \sum_{v \in S_n(\mathbf{w})} \deg^+(v) - 2(s - 1) + p \quad (3.3)$$

where, on the right-hand side, the first summand is the total number of non-trivial simple paths, the second summand is the number of non-trivial simple paths that are non-palindromic, and p is the number of special palindromes of length n (i.e., the number of trivial simple paths of order n that are palindromic). By observing that the number of special factors of length n is $2s - p$, we can simplify equation (3.3) to obtain the required equality (II) as follows:

$$\begin{aligned} \mathcal{P}(n) + \mathcal{P}(n + 1) &= \sum_{v \in S_n(\mathbf{w})} \deg^+(v) - (2s - p) + 2 \\ &= \sum_{v \in S_n(\mathbf{w})} (\deg^+(v) - 1) + 2 \\ &= \mathcal{C}(n + 1) - \mathcal{C}(n) + 2 \quad (\text{by (3.2)}). \end{aligned}$$

We observe, in particular, that any infinite word \mathbf{w} with $F(\mathbf{w})$ closed under reversal satisfies equality (II) if and only if any simple path between a special factor and its reversal is palindromic, and for each n , there are exactly $2(s - 1)$ non-trivial simple paths of order n that are non-palindromic. The latter condition says that, for all n , the super-reduced Rauzy graph $\Gamma_n''(\mathbf{w})$ contains exactly $s - 1$ edges (with each edge corresponding to a simple path and its reversal), and hence $\Gamma_n''(\mathbf{w})$ is a tree as it contains s vertices, $s - 1$ edges, and must be connected by the recurrence of \mathbf{w} (which follows from Proposition 2.2). More formally:

Proposition 3.5. *An infinite word \mathbf{w} with $F(\mathbf{w})$ closed under reversal satisfies equality (II) if and only if the following conditions hold:*

- 1) *any simple path between a special factor and its reversal is palindromic;*
- 2) *the super-reduced Rauzy graph $\Gamma_n''(\mathbf{w})$ is a tree for all n .*

Proof. Suppose \mathbf{w} is an infinite word with $F(\mathbf{w})$ closed under reversal. Then \mathbf{w} is recurrent by Proposition 2.2. We have already shown that conditions 1) and 2) imply that \mathbf{w} satisfies equality (II). Conversely, if at least one of conditions 1) and 2) does not hold, then $\mathcal{P}(n) + \mathcal{P}(n+1) < \mathcal{C}(n+1) - \mathcal{C}(n) + 2$ (by the arguments preceding this proposition), i.e., \mathbf{w} does not satisfy equality (II). \square

To complete the proof of “(I) \Rightarrow (II)”, it remains to show that any recurrent rich infinite word \mathbf{w} satisfies condition 2) of Proposition 3.5, since we have already shown that condition 1) holds for any such \mathbf{w} (using Lemma 3.3). The proof of the fact that \mathbf{w} satisfies condition 2) uses the following two lemmas (Lemmas 3.6–3.7).

Notation. Given two distinct special factors v, w of the same length n , we write $v \not\rightarrow w$ if there does not exist a directed edge from v to w in the reduced Rauzy graph $\Gamma'_n(\mathbf{w})$ (i.e., if there does not exist a simple path from v to w).

Lemma 3.6. *Suppose \mathbf{w} is a recurrent rich infinite word and let v, w be two distinct special factors of \mathbf{w} of the same length with $v \notin \{w, \tilde{w}\}$. If there exists a simple path P from v to w , then P is unique and there also exists a unique simple path from \tilde{w} to \tilde{v} (namely, the reversal of P). Moreover:*

- i) $v \not\rightarrow \tilde{w}$, and hence $w \not\rightarrow \tilde{v}$ (unless w is a palindrome);*
- ii) $\tilde{w} \not\rightarrow v$, and hence $\tilde{v} \not\rightarrow w$ (unless v is a palindrome);*
- iii) $w \not\rightarrow v$, and hence $\tilde{v} \not\rightarrow \tilde{w}$ (unless v and w are both palindromes).*

Proof. By closure under reversal (Remark 2.1), if there exists a simple path P from v to w , then the reversal of P is a simple path from \tilde{w} to \tilde{v} in the Rauzy graph of order $|v| = |w| = n$. To prove the uniqueness of P , let us suppose there exist two different simple paths P_1, P_2 from v to w in the Rauzy graph $\Gamma_n(\mathbf{w})$. Then

$$P_1 = vu_1 \cdots u_k w \quad \text{and} \quad P_2 = vz_1 \cdots z_\ell w \quad \text{for some } k, \ell \in \mathbb{N},$$

where $u_1, \dots, u_k, z_1, \dots, z_\ell$ are non-special factors of \mathbf{w} of length n and $u_i \neq z_i$ for some i . Note that either P_1 or P_2 (not both) may be of the form vw .

To keep the rest of the proof as simple as possible, we assume hereafter that neither v nor w is a palindrome; the arguments are similar, and in fact easier, in the cases when either v or w (or both) is a palindrome.

Consider a path Q of minimal length beginning with P_1 and ending with P_2 (in the Rauzy graph $\Gamma_n(\mathbf{w})$):

$$Q = P_1 \cdots P_2 = vu_1 \cdots u_k \underbrace{w \cdots v}_{Q_1} z_1 \cdots z_\ell w.$$

First we observe that Q contains \tilde{v} since any path from v to itself must pass through \tilde{v} , by Lemma 3.4. Moreover, the left-most \tilde{v} in Q must occur in the subpath Q_1 (since \tilde{v} is not equal to any of the non-special factors u_i, z_j and $\tilde{v} \neq w$). Therefore

$$Q = v \underbrace{u_1 \cdots u_k w \cdots \tilde{v}}_{Q_2} \cdots vz_1 \cdots z_\ell w$$

where the subpath Q_2 ends with the left-most \tilde{v} in the path Q . By Lemma 3.4, Q_2 is a path from v to \tilde{v} that does not contain v or \tilde{v} as an interior vertex. Thus, by Lemma 3.3, Q_2 is palindromic,

and hence Q_2 ends with the reversal of the path P_1 since it begins with P_1 . More explicitly:

$$Q = \underbrace{vu_1 \cdots u_k w}_{P_1} \cdots \underbrace{\tilde{w} \tilde{u}_k \cdots \tilde{u}_1 \tilde{v}}_{\tilde{P}_1} \overbrace{\cdots vz_1 \cdots z_\ell w}^{Q_3}.$$

We distinguish two cases.

Case 1: If the subpath Q_3 contains w as a terminal vertex only, then \tilde{w} is not an interior vertex of Q_3 by Lemma 3.4, and hence Q_3 is palindromic by Lemma 3.3. It follows that $k = \ell$ and $z_i = u_i$ for all $i = 1, \dots, k$. Thus $P_1 = P_2$; a contradiction.

Case 2: If the subpath Q_3 contains w as an interior vertex, then Q_3 first passes through w after taking the path \tilde{P}_1 (at the beginning) and before taking the path P_2 (at the end). Hence, by Lemma 3.3, Q_3 begins with a palindromic path from \tilde{w} to w that begins with \tilde{P}_1 and hence ends with P_1 . But then Q passes through the path P_1 at least twice before taking the path P_2 , contradicting the fact that Q is a path of minimal length beginning with P_1 and ending with P_2 .

Both cases lead to a contradiction; thus the simple path P from v to w is unique (and its reversal \tilde{P} is the unique simple path from \tilde{w} to \tilde{v}). It remains to show that conditions *i*)–*iii*) hold. As *ii*) is symmetric to *i*), we prove only that *i*) and *iii*) are satisfied. By what precedes, it suffices to consider paths in the reduced Rauzy graph $\Gamma'_n(\mathbf{w})$.

i): Arguing by contradiction, let us suppose that there exists a (unique) simple path from v to \tilde{w} , i.e., there exists a directed edge from v to \tilde{w} in the reduced Rauzy graph $\Gamma'_n(\mathbf{w})$. Then (from above) we know that there also exists a directed edge from w to \tilde{v} . Consider a shortest path Q in the reduced Rauzy graph $\Gamma'_n(\mathbf{w})$ beginning with $v\tilde{w}$ and ending with vw . By Lemma 3.4, any path from v to itself passes through \tilde{v} , so we may write

$$Q = \underbrace{v\tilde{w} \cdots \tilde{v}}_{Q_1} \cdots vw,$$

where the subpath Q_1 ends with the left-most \tilde{v} in the path Q . By Lemmas 3.3–3.4, the path $Q_1 = v\tilde{w} \cdots \tilde{v}$ is palindromic, and hence it ends with $w\tilde{v}$. So we have $Q = v\tilde{w} \cdots w\tilde{v} \cdots vw$; moreover, by Lemma 3.4, \tilde{w} must occur between the last two w 's shown here. In particular,

$$Q = v\tilde{w} \cdots \underbrace{w\tilde{v} \cdots \tilde{w}}_{Q_2} \cdots vw$$

where the subpath Q_2 contains \tilde{w} as a terminal vertex only. Thus, by Lemmas 3.3–3.4, the path $Q_2 = w\tilde{v} \cdots \tilde{w}$ is palindromic, and hence it ends with $v\tilde{w}$. But then Q ends with a shorter path of the form $v\tilde{w} \cdots vw$, contradicting the fact that Q is a path of minimal length beginning with $v\tilde{w}$ and ending with vw .

iii): Again, the proof proceeds by contradiction. Suppose there exists a (unique) simple path from w to v . Consider a shortest path Z in the reduced Rauzy graph $\Gamma'_n(\mathbf{w})$ beginning with wv and ending with vw . By Lemma 3.4, the path Z must pass through \tilde{w} ; thus

$$Z = \underbrace{wv \cdots \tilde{w}}_{Z_1} \cdots vw.$$

where the subpath Z_1 ends with the left-most \tilde{w} in the path Z . Now it follows from Lemmas 3.3–3.4 that the subpath Z_1 is palindromic, and hence Z_1 must end with $\tilde{v}\tilde{w}$. So we may write

$$Z = wv \cdots \underbrace{\tilde{v}\tilde{w} \cdots v}_{Z_2} w.$$

If the subpath Z_2 contains v as a terminal vertex only, then neither v nor \tilde{v} is an interior vertex of Z_2 by Lemma 3.4. Thus Z_2 is palindromic by Lemma 3.3, and hence Z_2 ends with wv . But then the path Z ends with the path wvw , which is impossible by Lemma 3.4. Thus, the subpath Z_2 must pass through v at an earlier point, and hence we have $Z_2 = \tilde{v}\tilde{w} \cdots v \cdots v$. In particular, the path Z_2 begins with a palindromic subpath of the form $\tilde{v}\tilde{w} \cdots wv$, by Lemma 3.3. But then the path Z ends with a shorter path from wv to wv , contradicting the minimality of Z . \square

Notation. For a finite word v , let v^ϵ represent either v or \tilde{v} and set $v^{-\epsilon} := \tilde{v}^\epsilon$.

Lemma 3.7. *Let \mathbf{w} be a recurrent rich infinite word. For fixed $n \in \mathbb{N}_+$, suppose the super-reduced Rauzy graph $\Gamma_n''(\mathbf{w})$ contains at least three distinct vertices: $[v_1], [v_2], \dots, [v_s]$, $s \geq 3$. Then, for each k with $3 \leq k \leq s$, the reduced Rauzy graph $\Gamma_n'(\mathbf{w})$ contains a path from v_1 to $v_k^{\epsilon_k}$ of the form:*

$$v_1 v_2^{\epsilon_2} \cdots v_2 v_3^{\epsilon_3} \cdots v_{k-2} v_{k-1}^{\epsilon_{k-1}} \cdots v_{k-1} v_k^{\epsilon_k},$$

where for all $i = 2, \dots, k-1$, the subpath $v_i^{\epsilon_i} \cdots v_i$ (which may consist of only the single vertex $v_i^{\epsilon_i}$) does not contain v_j, \tilde{v}_j for all j with $1 \leq j \leq k$, $j \neq i$.

Proof. We use induction on k and employ similar reasoning to the proof of Lemma 3.6.

First consider the case $k = 3$. Recurrence implies that $\Gamma_n'(\mathbf{w})$ is connected, so we may assume without loss of generality that $\Gamma_n'(\mathbf{w})$ contains a directed edge from v_1 to $v_2^{\epsilon_2}$, a directed edge from v_2 to $v_3^{\epsilon_3}$, and a path from $v_2^{\epsilon_2}$ to v_2 . That is, $\Gamma_n'(\mathbf{w})$ contains a path beginning with $v_1 v_2^{\epsilon_2}$ and ending with $v_2 v_3^{\epsilon_3}$. Consider such a path of minimal length:

$$Q = v_1 v_2^{\epsilon_2} \cdots v_2 v_3^{\epsilon_3}.$$

To prove the claim for $k = 3$, we show that none of the special factors $v_1, \tilde{v}_1, v_3, \tilde{v}_3$ are interior vertices of Q . If $v_2^{\epsilon_2} = v_2$, then $Q = v_1 v_2 v_3^{\epsilon_3}$ (by minimality) and we are done. So let us assume that $v_2^{\epsilon_2} = \tilde{v}_2 \neq v_2$.

Observe that if v_1 is an interior vertex of Q , then \tilde{v}_1 must be an interior vertex of Q since any path from v_1 to itself must contain \tilde{v}_1 , by Lemma 3.4. Similarly, if $v_3^{\epsilon_3}$ is an interior vertex of Q , then $v_3^{-\epsilon_3}$ is an interior vertex of Q . Therefore it suffices to show that \tilde{v}_1 and $v_3^{-\epsilon_3}$ are not interior vertices of Q . We prove this fact only for \tilde{v}_1 as the proof is similar for $v_3^{-\epsilon_3}$.

Arguing by contradiction, suppose \tilde{v}_1 is an interior vertex of Q . Then Q begins with a palindromic path from $v_1 \tilde{v}_2$ to \tilde{v}_1 (by Lemmas 3.3–3.4), and this palindromic path clearly ends with $v_2 \tilde{v}_1$. Hence

$$Q = v_1 \tilde{v}_2 \cdots \underbrace{v_2 \tilde{v}_1 \cdots v_2}_{Q'} v_3^{\epsilon_3}$$

where the subpath Q' begins with a palindromic path from $v_2 \tilde{v}_1$ to \tilde{v}_2 (by Lemmas 3.3–3.4), and this palindromic path clearly ends with $v_1 \tilde{v}_2$. But then the path Q ends with a shorter path from $v_1 \tilde{v}_2$ to $v_2 v_3^{\epsilon_3}$, contradicting the minimality of Q . Thus the lemma holds for $k = 3$.

Now suppose $4 \leq k \leq s$ and assume the claim holds for $k - 1$. Since $\Gamma'_n(\mathbf{w})$ is connected, it contains a path beginning with $v_1 v_2^{\epsilon_2} \cdots v_2 v_3^{\epsilon_3} \cdots v_{k-2} v_{k-1}^{\epsilon_{k-1}}$ and ending with $v_{k-1} v_k^{\epsilon_k}$ (where the former path satisfies the conditions of the lemma). Consider such a path of minimal length:

$$Z = \underbrace{v_1 v_2^{\epsilon_2} \cdots v_2 v_3^{\epsilon_3} \cdots v_{k-2}}_{Z_1} \underbrace{v_{k-1}^{\epsilon_{k-1}} \cdots v_{k-1} v_k^{\epsilon_k}}_{Z_2} \quad (3.4)$$

where for all $i = 2, \dots, k - 2$, the subpath $v_i^{\epsilon_i} \cdots v_i$ (which may consist of only the single vertex $v_i^{\epsilon_i}$) does not contain v_j, \tilde{v}_j for all j with $1 \leq j \leq k - 1, j \neq i$. To prove the induction step, we show that the path Z satisfies the following two conditions:

i) the subpath Z_1 contains neither v_k nor \tilde{v}_k ;

ii) the subpath $Z_2 = v_{k-1}^{\epsilon_{k-1}} \cdots v_{k-1}$ does not contain v_j, \tilde{v}_j for all j with $1 \leq j \leq k, j \neq k - 1$.

First suppose that condition *i)* is not satisfied, i.e., Z_1 contains v_k or \tilde{v}_k . Without loss of generality we assume that v_k is the right-most of the vertices v_k, \tilde{v}_k appearing in Z_1 .

Case 1: Suppose $v_k^{\epsilon_k} = v_k \neq \tilde{v}_k$. Then Z ends with a path from v_k to itself, which must pass through \tilde{v}_k by Lemma 3.4; moreover, \tilde{v}_k must be an interior vertex of Z_2 (by the choice of v_k). Thus, by Lemmas 3.3–3.4, $Z_2 v_k$ (and hence Z) ends with a palindromic path from \tilde{v}_k to $v_{k-1} v_k$. Hence Z_2 contains $\tilde{v}_k \tilde{v}_{k-1}$, and we have:

$$Z_2 v_k = \underbrace{v_{k-1}^{\epsilon_{k-1}} \cdots \tilde{v}_k \tilde{v}_{k-1}}_{Z_3} \cdots v_{k-1} v_k$$

where the subpath Z_3 ends with a palindromic path from v_{k-1} to $\tilde{v}_k \tilde{v}_{k-1}$ (by Lemmas 3.3–3.4); thus Z_3 contains $v_{k-1} v_k$. But then Z begins with a shorter path from Z_1 to $v_{k-1} v_k^{\epsilon_k}$, contradicting the minimality of Z .

Case 2: Suppose $v_k^{\epsilon_k} = \tilde{v}_k$. Then the path $Z (= Z_1 Z_2 \tilde{v}_k)$ ends with a path of the form:

$$Z_4 = v_k \underbrace{\cdots \cdots}_{\text{no } v_k, \tilde{v}_k} Z_2 \tilde{v}_k.$$

If v_k or \tilde{v}_k is an interior vertex of Z_2 , then we reach a contradiction using the same arguments as in Case 1. On the other hand, if neither v_k nor \tilde{v}_k is an interior vertex of Z_2 , then Z_4 is palindromic by Lemma 3.3. So the path Z_4 begins with $v_k \tilde{v}_{k-1}$ since it ends with $v_{k-1} \tilde{v}_k$. But then \tilde{v}_{k-1} is an interior vertex of Z_1 , a contradiction.

Thus the path Z satisfies condition *i)*. In proving this fact, we have also shown that v_k, \tilde{v}_k are not interior vertices of Z_2 . It remains to show that the subpath Z_2 does not contain v_j, \tilde{v}_j for all j with $1 \leq j \leq k - 2$ (and hence Z satisfies condition *ii)*). We prove only that Z_2 does not contain \tilde{v}_1 or \tilde{v}_1 since the proof is similar when considering other v_j, \tilde{v}_j .

Suppose on the contrary that Z_2 contains v_1 or \tilde{v}_1 . Then, by Lemmas 3.3–3.4, Z begins with a palindromic path from v_1 to \tilde{v}_1 , and this palindromic path begins with $Y = Z_1 v_{k-1}^{\epsilon_{k-1}}$ (and hence ends with \tilde{Y}) by the conditions on Z under the induction hypothesis. More explicitly, we have:

$$Z = \underbrace{v_1 v_2^{\epsilon_2} \cdots v_{k-2} v_{k-1}^{\epsilon_{k-1}}}_{Y} \overbrace{v_{k-1}^{-\epsilon_{k-1}} \tilde{v}_{k-2} \cdots v_2^{-\epsilon_2} \tilde{v}_1}^{\text{palindromic}} \underbrace{\cdots v_{k-1} v_k^{\epsilon_k}}_{Z_5}.$$

Hence, as v_{k-1} and \tilde{v}_{k-1} are not interior vertices of Y (by the induction hypothesis), the subpath $\tilde{Y}Z_5$ begins with a palindromic path from $v_{k-1}^{-\epsilon_k}$ to $v_{k-1}^{\epsilon_{k-1}}$, and this palindromic path begins with \tilde{Y} (and hence ends with Y), by Lemmas 3.3–3.4. But then Z ends with a shorter path from Y to $v_{k-1}v_k^{\epsilon_k}$, contradicting the minimality of Z .

We conclude that the subpath $Z_2 = v_{k-1}^{\epsilon_{k-1}} \cdots v_{k-1}$ does not contain v_j, \tilde{v}_j for all j with $1 \leq j \leq k$, $j \neq i$ (i.e., the path Z satisfies condition *ii*), and the proof is thus complete. \square

Lemma 3.8. *Suppose \mathbf{w} is a recurrent rich infinite word. Then the super-reduced Rauzy graph $\Gamma_n''(\mathbf{w})$ is a tree for all $n \in \mathbb{N}_+$.*

Proof. First recall that for all n , $\Gamma_n''(\mathbf{w})$ is connected (by the recurrence property of \mathbf{w}). Moreover, Lemma 3.6 tells us that if two distinct vertices in $\Gamma_n''(\mathbf{w})$ are joined by an edge, then this edge is unique (and corresponds to a simple path and its reversal). It remains to show that $\Gamma_n''(\mathbf{w})$ does not contain any *cycle* (i.e., does not contain a chain linking a vertex with itself).

Suppose on the contrary that $\Gamma_n''(\mathbf{w})$ contains a cycle for some n . Then $\Gamma_n''(\mathbf{w})$ must contain at least three distinct vertices: $[v_1], [v_2], \dots, [v_s]$, $s \geq 3$, and a cycle of the following form:

$$[v_1] - [v_2] - \cdots - [v_k] - [v_1] \quad \text{for some } k \text{ with } 3 \leq k \leq s. \quad (3.5)$$

We thus deduce from Lemma 3.7 that the reduced Rauzy graph $\Gamma_n'(\mathbf{w})$ contains a path from v_1 to $v_1^{\epsilon_1}$ of the form:

$$P = v_1 v_2^{\epsilon_2} \cdots v_2 v_3^{\epsilon_3} \cdots v_{k-2} v_{k-1}^{\epsilon_{k-1}} \cdots v_{k-1} v_k^{\epsilon_k} \cdots v_k v_1^{\epsilon_1},$$

where for all $i = 2, \dots, k$, the subpath $v_i^{\epsilon_i} \cdots v_i$ (which may consist of only the single vertex $v_i^{\epsilon_i}$) does not contain v_j, \tilde{v}_j for all j with $1 \leq j \leq k$, $j \neq i$. (Note that P corresponds to the cycle given in (3.5).)

First suppose that v_1 is a palindrome. In this case, as neither v_1 nor \tilde{v}_1 is an interior vertex of P , it must be a palindromic path by Lemma 3.3. But then $v_k = v_2^{-\epsilon_2}$, a contradiction (as $k \geq 3$).

Now suppose that v_1 is not a palindrome. If $v_1^{\epsilon_1} = \tilde{v}_1$, then we deduce (as above, using Lemma 3.3) that the path P must be palindromic, yielding a contradiction. On the other hand, if $v_1^{\epsilon_1} = v_1$, then, by Lemma 3.4, the path P must pass through \tilde{v}_1 , a contradiction.

Thus $\Gamma_n''(\mathbf{w})$ is a tree. \square

This concludes our proof of the “(I) \Rightarrow (II)” part of Theorem 1.1.

3.2 (II) implies (I)

Conversely, suppose \mathbf{w} is an infinite word with $F(\mathbf{w})$ closed under reversal and satisfying equality (II). Then \mathbf{w} satisfies conditions 1) and 2) of Proposition 3.5.

Now, arguing by contradiction, suppose \mathbf{w} does not satisfy property (I) (i.e., \mathbf{w} is not rich). Then there exists a palindromic factor p that has a non-palindromic complete return u in \mathbf{w} ; in particular, we have $u = pqavb\tilde{q}p$ for some words q, v (possibly empty) and letters a, b , with $a \neq b$. So the words $pqa, b\tilde{q}p$ and their reversals $a\tilde{q}p, pqb$ are factors of \mathbf{w} . Thus pq (resp. $\tilde{q}p$) is a right-special (resp. left-special) factor of \mathbf{w} . Hence, if u does not contain any other special factors, then u forms the label of a non-palindromic simple path beginning with pq and ending with $\tilde{q}p$. But this contradicts condition 1) of Proposition 3.5. Therefore u must contain other special factors of length $n := |pq|$, besides pq and $\tilde{q}p$. In particular, u begins with the label of a simple path of order n beginning with pq and ending with another special factor s_1 of length n . Similarly, u ends with the label of a simple path of order n beginning with a special factor s_2 of length n and ending with $\tilde{q}p$. Moreover, since u is a complete return to p , neither s_1 nor s_2 is equal to pq or $\tilde{q}p$ (otherwise

p occurs as an interior factor of u). Thus, in the super-reduced Rauzy graph $\Gamma_n''(\mathbf{w})$, there is an edge between the vertex $[pq]$ and each of the vertices $[s_1]$ and $[s_2]$. In particular, there exists a path of the form: $[s_1]—[pq]—[s_2]$. Furthermore, as u contains a factor that begins with s_1 and ends with s_2 and contains no occurrence of pq or $\tilde{q}p$, there also exists a chain (or possibly just an edge) linking $[s_1]$ and $[s_2]$ that does not contain the vertex $[pq]$. Thus, if $\{s_1, \tilde{s}_1\} \neq \{s_2, \tilde{s}_2\}$, then we see that $\Gamma_n''(\mathbf{w})$ contains a cycle, contradicting condition 2) of Proposition 3.5. On the other hand, if $\{s_1, \tilde{s}_1\} = \{s_2, \tilde{s}_2\}$, then there are at least two edges joining the vertices $[s_1]$ and $[pq]$. Indeed, there exists a simple path P_1 from pq to s_1 and there also exists a simple path P_2 either from s_1 to $\tilde{q}p$ or from \tilde{s}_1 to $\tilde{q}p$. By closure under reversal, the reversals \tilde{P}_1, \tilde{P}_2 of the respective simple paths P_1, P_2 also exist. Moreover, none of these four simple paths coincide. Certainly, $P_1 \neq P_2$, $P_1 \neq \tilde{P}_1$, and $P_2 \neq \tilde{P}_2$ as neither s_1 nor \tilde{s}_1 is equal to pq or $\tilde{q}p$, and $P_1 \neq \tilde{P}_2$ as the second vertex in P_1 ends with the letter a , whereas the second vertex in the path \tilde{P}_2 ends with the letter $b \neq a$. So $\Gamma_n''(\mathbf{w})$ is not a tree, contradicting condition 2) of Proposition 3.5. This concludes our proof of Theorem 1.1. \square

4 A few consequences and remarks

From Theorem 1.1, we easily deduce that property (I) is equivalent to equality (II) for any uniformly recurrent infinite word. Indeed, equality (II) implies the existence of arbitrarily long palindromes since $\mathcal{P}(n) + \mathcal{P}(n+1) \geq 2$ for all n , so together with uniform recurrence one can readily show that factors are closed under reversal; hence property (I) holds by Theorem 1.1. Conversely, richness (property (I)) together with uniform recurrence implies closure under reversal by Remark 2.1, and hence equality (II) holds.

Question: *In the statement of Theorem 1.1, can the hypothesis of factors being closed under reversal be replaced by the weaker hypothesis of recurrence?*

As above, it follows directly from Theorem 1.1 and Remark 2.1 that for any recurrent infinite word \mathbf{w} , if \mathbf{w} satisfies property (I) (i.e., if \mathbf{w} is rich, and hence has factors closed under reversal), then equality (II) holds. However, to prove the converse using our methods, one would need to know that any recurrent infinite word satisfying equality (II) has factors closed under reversal. We could not find a proof of this claim nor could we find a counter-example. Let us point out that whilst uniform recurrence and the existence of arbitrarily long palindromes imply closure under reversal, this is not true in the case of recurrence only. For instance, consider the following infinite word:

$$\mathbf{s} = bca^2bca^3bca^2bca^4bca^2bca^3bca^2bca^5bc \dots,$$

which is the limit as n goes to infinity of the sequence $(s_n)_{n \geq 1}$ of finite words defined by:

$$s_1 = bc \quad \text{and} \quad s_n = s_{n-1}a^n s_{n-1} \quad \text{for } n > 1.$$

This infinite word is clearly recurrent (but not uniformly recurrent) and contains arbitrarily long palindromes, but its set of factors is not closed under reversal. (Note that \mathbf{s} is not rich and does not satisfy equality (II).) If one could show that recurrence together with equality (II) implies arbitrarily long palindromic prefixes, this would be enough to prove that factors are closed under reversal.

In the context of finite words w , the hypothesis of factors being closed under reversal can be replaced by the requirement that w is a palindrome. Indeed, all we really need is the super-reduced Rauzy graph to be connected, which is true for palindromes.

Theorem 4.1. *For any palindrome w , the following properties are equivalent:*

- i) w contains $|w| + 1$ distinct palindromes;*
- ii) all complete returns to palindromes in w are palindromes;*
- iii) $\mathcal{P}(i) + \mathcal{P}(i + 1) = \mathcal{C}(i + 1) - \mathcal{C}(i) + 2$ for all i with $0 \leq i \leq |w|$.* □

We now prove two easy consequences of Theorem 1.1.

Corollary 4.2. *Suppose \mathbf{w} is a recurrent rich infinite word. Then the following properties hold.*

- i) \mathbf{w} is (purely) periodic if and only if $\mathcal{P}(n) + \mathcal{P}(n + 1) = 2$ for some n .*
- ii) $(\mathcal{P}(n))_{n \geq 1}$ is eventually periodic with period 2 if and only if there exist non-negative integers K, L, N such that $\mathcal{C}(n) = Kn + L$ for all $n \geq N$.*

Proof. Suppose \mathbf{w} is a recurrent rich infinite word. Then $\mathcal{P}(n) + \mathcal{P}(n + 1) = \mathcal{C}(n + 1) - \mathcal{C}(n) + 2$ for all n , by Theorem 1.1 and Remark 2.1.

i): If $\mathcal{P}(n) + \mathcal{P}(n + 1) = 2$ for some n , then $\mathcal{C}(n + 1) = \mathcal{C}(n)$, and hence \mathbf{w} is eventually periodic; in particular, \mathbf{w} must be (purely) periodic as it is recurrent. Conversely, if \mathbf{w} is periodic, then $\mathcal{C}(n + 1) = \mathcal{C}(n)$ for some n , and hence $\mathcal{P}(n) + \mathcal{P}(n + 1) = 2$.

ii): The condition on $\mathcal{C}(n)$ implies that for all $n \geq N$, $\mathcal{C}(n + 1) - \mathcal{C}(n) = K$, and hence $\mathcal{P}(n) + \mathcal{P}(n + 1) = K + 2 = \mathcal{P}(n + 1) + \mathcal{P}(n + 2)$. Thus $\mathcal{P}(n) = \mathcal{P}(n + 2)$ for all $n \geq N$. Conversely, suppose $(\mathcal{P}(n))_{n \geq 1}$ is eventually periodic with period 2. Then there exists a non-negative integer N such that $\mathcal{P}(n) = \mathcal{P}(n + 2)$ for all $n \geq N$. Hence, for all $n \geq N$, $\mathcal{P}(n) + \mathcal{P}(n + 1) = \mathcal{C}(n + 1) - \mathcal{C}(n) + 2 = \mathcal{P}(n + 1) + \mathcal{P}(n + 2) = M \geq 2$. Therefore $\mathcal{C}(n + 1) - \mathcal{C}(n) = M - 2$ for all $n \geq N$. □

Remark 4.3. Item *ii)* of the above corollary can be compared with a result of J. Cassaigne [8], who proved that if $\mathcal{C}(n)$ has linear growth, then $\mathcal{C}(n + 1) - \mathcal{C}(n)$ is bounded.

Remark 4.4. In [5], Balázi *et al.* remarked: “According to our knowledge, all known examples of infinite words which satisfy the equality $\mathcal{P}(n) + \mathcal{P}(n + 1) = \mathcal{C}(n + 1) - \mathcal{C}(n) + 2$ for all $n \in \mathbb{N}$ have sublinear factor complexity.” Actually, there do exist recurrent rich infinite words with non-sublinear complexity. For instance, the following example from [19]: $abab^2abab^3abab^2abab^4abab^2abab^3abab^2abab^5 \dots$ (which is the fixed point of the morphism: $a \mapsto abab$, $b \mapsto b$) is a recurrent rich infinite word and its complexity $\mathcal{C}(n)$ grows quadratically with n . Another example that was indicated to us by J. Cassaigne is the fixed point of $a \mapsto aab$, $b \mapsto b$:

$$aabaabbaabaabbaabaabbaabaabbbbaabaabbaabaabbaabaabbaabaabbaabaabbbbbb \dots$$

It is a recurrent rich infinite word and its complexity is equivalent to $n^2/2$. More precisely, $\mathcal{P}(n) + \mathcal{P}(n + 1) - 2 = \mathcal{C}(n + 1) - \mathcal{C}(n) = n + 1 - \#\{k > 0 \mid 2^k + k - 2 < n\}$.

In [10], X. Droubay *et al.* showed that the family of *episturmian words* (e.g., see [10, 21, 18]), which includes the well-known *Sturmian words*, comprises a special class of uniformly recurrent rich infinite words. Specifically, they proved that if an infinite word \mathbf{w} is episturmian, then any factor u of \mathbf{w} contains exactly $|u| + 1$ distinct palindromic factors (see [10, Cor. 2]). An alternative proof of the richness of episturmian words can be found in the paper [3] where the fourth author, together with V. Anne and I. Zorca, proved that for episturmian words, all complete returns to palindromes are palindromes. (A shorter proof of this fact is also given in [7].) More recently,

P. Baláži *et al.* [5] showed that all *strict* episturmian words (i.e., *Arnoux-Rauzy sequences* [4, 23]) satisfy $\mathcal{P}(n) + \mathcal{P}(n + 1) = \mathcal{C}(n + 1) - \mathcal{C}(n) + 2$ for all n . This fact, together with Theorem 1.1, provides yet another proof that *all* episturmian words are rich (since any factor of an episturmian word is a factor of some strict episturmian word).

Sturmian words are exactly the aperiodic episturmian words over a 2-letter alphabet. They have complexity $n + 1$ for each n and are characterized by their palindromic complexity: any Sturmian word has $\mathcal{P}(n) = 1$ whenever n is even and $\mathcal{P}(n) = 2$ whenever n is odd (see [11]). From these observations, one can readily check that Sturmian words satisfy equality (II) (and hence they are rich).

We can now say even more: the set of factors of all Sturmian words satisfies equality (II). To show this, we first recall that F. Mignosi [22] proved that, for any $n \geq 0$, the number $c(n)$ of *finite Sturmian words* of length n is given by

$$c(n) = 1 + \sum_{i=1}^n (n + 1 - i)\phi(i),$$

where ϕ is *Euler's totient function*. More recently, in [9], the second author together with A. de Luca proved that for any $n \geq 0$, the number $p(n)$ of Sturmian palindromes of length n is given by

$$p(n) = 1 + \sum_{i=0}^{\lceil n/2 \rceil - 1} \phi(n - 2i).$$

Equivalently, for any $n \geq 0$,

$$p(2n) = 1 + \sum_{i=1}^n \phi(2i) \quad \text{and} \quad p(2n + 1) = 1 + \sum_{i=0}^n \phi(2i + 1).$$

Thus, for all $n \geq 0$,

$$p(2n) + p(2n + 1) = 2 + \sum_{i=1}^n \{\phi(2i) + \phi(2i + 1)\} + 2 = \sum_{i=1}^{2n+1} \phi(i) + 2,$$

and

$$\begin{aligned} c(2n + 1) - c(2n) + 2 &= \sum_{i=1}^{2n+1} (2n + 2 - i)\phi(i) - \sum_{i=1}^{2n} (2n + 1 - i)\phi(i) + 2 \\ &= \phi(2n + 1) + \sum_{i=1}^{2n} \phi(i) + 2 \\ &= \sum_{i=1}^{2n+1} \phi(i) + 2 = p(2n) + p(2n + 1). \end{aligned}$$

From this point of view, it would be interesting to count for instance the number of all binary rich words of length n for each n .

Acknowledgements. The authors would like to thank Jacques Justin for helpful comments and suggestions on a preliminary version of this paper. The first three authors would also like to acknowledge the hospitality of the Department of Mathematics at the University of North Texas where this work was done.

References

- [1] J.-P. Allouche, M. Baake, J. Cassaigne, D. Damanik, Palindrome complexity, *Theoret. Comput. Sci.* 292 (2003) 9–31.
- [2] P. Ambrož, C. Frougny, Z. Masáková, E. Pelantová, Palindromic complexity of infinite words associated with simple Parry numbers, *Ann. Inst. Fourier (Grenoble)* 56 (2006) 2131–2160.
- [3] V. Anne, L.Q. Zamboni, I. Zorca, Palindromes and pseudo-palindromes in episturmian and pseudo-palindromic infinite words, in: *Proceedings of the Fifth International Conference on Words* (Montréal, Canada), September 13–17, 2005. *Publications du LaCIM* 36 (2005) 91–100.
- [4] P. Arnoux, G. Rauzy, Représentation géométrique de suites de complexité $2n + 1$, *Bull. Soc. Math. France* 119 (1991) 199–215.
- [5] P. Baláži, Z. Masáková, E. Pelantová, Factor versus palindromic complexity of uniformly recurrent infinite words, *Theoret. Comput. Sci.* 380 (2007) 266–275.
- [6] S. Brlek, S. Hamel, M. Nivat, C. Reutenauer, On the palindromic complexity of infinite words, *Internat. J. Found. Comput. Sci.* 15 (2004) 293–306.
- [7] M. Bucci, A. de Luca, A. De Luca, L.Q. Zamboni, On some problems related to palindromic closure, *Theoret. Inform. Appl.* (in press), doi:10.1051/ita:2007064.
- [8] J. Cassaigne, Special factors of sequences with linear subword complexity, in: *Developments in Language Theory II*, World Scientific, Singapore, 1996, pp. 25–34.
- [9] A. de Luca, A. De Luca, Combinatorial properties of Sturmian palindromes, *Internat. J. Found. Comput. Sci.* 17 (2006) 557–573.
- [10] X. Droubay, J. Justin, G. Pirillo, Episturmian words and some constructions of de Luca and Rauzy, *Theoret. Comput. Sci.* 255 (2001) 539–553.
- [11] X. Droubay, G. Pirillo, Palindromes and Sturmian words, *Theoret. Comput. Sci.* 223 (1999) 73–85.
- [12] F. Durand, A characterization of substitutive sequences using return words, *Discrete Math.* 179 (1998) 89–101.
- [13] F. Durand, A generalization of Cobham’s theorem, *Theory Comput. Syst.* 31 (1998) 169–185.
- [14] F. Durand, Linearly recurrent subshifts have a finite number of non-periodic subshift factors, *Ergodic Theory Dynam. Systems* 19 (1999) 953–993.
- [15] S. Ferenczi, C. Mauduit, A. Nogueira, Substitutional dynamical systems: algebraic characterization of eigenvalues, *Ann. Sci. École Norm. Sup.* 29 (1995) 519–533.
- [16] S. Fischler, Palindromic prefixes and episturmian words, *J. Combin. Theory Ser. A* 113 (2006) 1281–1304.
- [17] S. Fischler, Palindromic prefixes and diophantine approximation, *Monatsh. Math.* 151 (2007) 11–37.
- [18] A. Glen, J. Justin, Episturmian words: a survey, Preprint, 2007, arXiv:0801.1655.

- [19] A. Glen, J. Justin, S. Widmer, L.Q. Zamboni, Palindromic richness, *European J. Combin.* (in press), doi:10.1016/j.ejc.2008.04.006.
- [20] C. Holton, L.Q. Zamboni, Descendants of primitive substitutions, *Theory Comput. Syst.* 32 (1999) 133–157.
- [21] J. Justin, G. Pirillo, Episturmian words and episturmian morphisms, *Theoret. Comput. Sci.* 276 (2002) 281–313.
- [22] F. Mignosi, On the number of factors of Sturmian words, *Theoret. Comput. Sci.* 82 (1991) 71–84.
- [23] G. Rauzy, Suites à termes dans un alphabet fini, in: *Sémin. Théorie des Nombres*, Exp. No. 25, pp. 16, Univ. Bordeaux I, Talence, 1982–1983.
- [24] A. Siegel, Pure discrete spectrum dynamical systems and periodic tiling associated with a substitution, *Ann. Inst. Fourier (Grenoble)* 54 (2004) 341–381.