THE EFFECT OF SURFACE TENSION ON FREE-SURFACE FLOW INDUCED BY A POINT SINK

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Abstract

The steady, axisymmetric flow induced by a point sink (or source) submerged in an inviscid fluid of infinite depth is computed and the resulting deformation of the free surface is obtained. The effect of surface tension on the free surface is determined and is the new component of this work. The maximum Froude numbers at which steady solutions exist are computed. It is found that the determining factor in reaching the critical flow changes as more surface tension is included. If there is zero or a very small amount of surface tension, the limiting factor appears to be the formation of small wavelets on the free surface; but, as the surface tension increases, this is replaced by a tendency for the lowest point on the free surface to descend sharply as the Froude number is increased.

2010 *Mathematics subject classification*: 0102.

Keywords and phrases: selective withdrawal, point sink, axisymmetric flow, free surface, surface tension.

1. Introduction

Studies of the manner of withdrawal of water from water storages are important to aid in understanding the nature of the flow and to enable better management of water resources, especially in dry climatic zones. Much work has been done in dealing with stratified water bodies and different geometries. Here we consider the very simple case of a single, uniform body of water with axisymmetric withdrawal through a submerged point outlet. In this work, we include the effect of surface tension, the addition of which not only adds an extra physical dimension to the problem bringing into context the earlier work without surface tension [3, 26], but also has a regularizing effect on the calculations.

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A recent paper [11] showed that the maximum flow rate at which steady solutions to this problem exist was significantly over-estimated in the earlier work [3, 26]. More accurate solutions indicate that as soon as small waves begin to form on the free surface, the steady solutions fail. These results are consistent with unsteady simulations [19].

It is reasonable to assume that the effect of surface tension will be to dampen these waves, thus allowing the flow to exist at higher flow rates. Indeed, the results herein show this to be the case. As the amount of surface tension increases, the maximum flow rate for steady flow increases.

Early work on withdrawal of water through a line sink (two-dimensional flow) [2, 18, 22] was focused on a cusp shape that formed on the free surface and was believed to represent the critical transition (the so-called *critical drawdown* in the case of a sink) between a single-layer flow and a two-layer flow, in which both layers are being withdrawn; for example, air is also drawn into the sink in the case of an air–water interface. Peregrine [17] was the first to compute single-layer solutions at low flow rates that had a stagnation point on the free surface directly above the sink. In two dimensions, it is possible to show that for ideal fluid flow the only possible solutions are either a cusp shape or a stagnation point flow so long as the flow is restricted to a single layer [23].

Cusped solutions in more general geometries for a line sink were found [7, 23, 25], while Vanden Broeck et al. [16, 27] and Forbes and Hocking [4, 9] computed further solutions with a stagnation point on the free surface. Vanden Broeck and Keller [25] found many solutions with a cusp shape in a fluid of finite depth, but in all cases the situation with a fluid of infinite depth gave cusp solutions at a unique flow rate [7, 23]. A gap was found between the maximum flow rate at which solutions with a stagnation point and solutions with a cusp were obtained. When surface tension was included in the problem [4, 9], a fold in the parameter space led to multiple solutions and a maximum flow rate beyond which no steady solutions with a stagnation point could be obtained.

Some of this uncertainty regarding critical drawdown of the free surface for flow into a line sink was resolved by Hocking and Forbes [8, 10], who computed steady, two-layer supercritical flows. They found that as the (supercritical) flow rate was reduced, the limiting two-layer flow approached the single-layer cusped-surface solutions, seeming to verify that the cusped solutions are indeed the critical transition flows.

In the corresponding three-dimensional axisymmetric case of flow into a point sink, attempts to compute cusp-like solutions [5] met with limited success, with such solutions only found over a narrow range in parameter space. Forbes and others [3, 6, 12, 26] considered the steady, axisymmetric flow into a point sink submerged beneath a free surface in semi-infinite and finite-depth fluid domains. They found that there is a maximum flow rate beyond which steady solutions do not exist, but no evidence of drawndown surfaces was obtained.

Recent work by Hocking and Zhang [13] in the analogous flows into a point sink in porous media have provided solutions in the steady, supercritical flow regime and shown the relationship between single-layer and two-layer flows, but such solutions have not been obtained in the surface-water case.

In the three-dimensional unsteady flow due to a point sink, Tyvand [24] performed an analysis for small time, while Xue and Yue [28] and Lubin and Springer [15] did numerical calculations. More recently, Stokes et al. [20, 21] performed detailed computations to determine the critical drawdown values for withdrawal into point and ring sinks. Their work showed that it is not simply the case that a constant flow rate would progress through some unsteady flow history before settling into a steady state or to drawdown. In fact, it has been shown that there are a number of possible outcomes including nonuniqueness in parameter space, jet and splash formation, and multiple critical drawdown values, depending on flow history.

It is, therefore, still of interest to consider the steady-flow problem. Our concern here is with the extension of the existing steady-flow results for a point sink to consider the effect of surface tension. In the two-dimensional case (line sink [4, 9]), this was found to provide considerable information about the flows including nonuniqueness in the solution space. Holmes and Hocking [14] recently showed that this nonuniqueness also occurs when the submerged sink is situated in a flowing stream.

In the current work, we find that for very small surface tension the results are much the same as the case with no surface tension. However, with even moderate surface tension the limiting steady solutions have a large dip in the free surface near to (but not directly above) the outlet. This dip increases in depth sharply as the maximum Froude number is approached. The presence of surface tension stabilizes the flow, so that steady solutions exist at higher flow rates.

2. Problem formulation

Consider the steady, irrotational, axisymmetric flow of an inviscid, incompressible fluid beneath a free surface. The flow is driven by a point sink of strength m situated at a depth H beneath the undisturbed level of the free surface. Under these assumptions, the problem can be formulated in terms of a velocity potential $\phi(r, z)$, so that the radial velocity component is $u = \phi_r$ and the vertical velocity component is $v = \phi_z$, where r is a radial coordinate centred on the location of the point sink and z is the vertical coordinate with $z = 0$ corresponding to the level of the free surface if there is no flow. The sink sits at a depth of $z = -H$ and the surface is subject to surface tension T, while the fluid is semi-infinite in extent (not bounded below).

Nondimensionalizing the velocity and length with respect to m/H^2 and H, respectively, where the quantity m is the total flux from the full point sink, the problem is to solve

$$\nabla^2\phi(r, z) = 0, \quad z < \eta(r), \quad (r, z) \neq (0, -1), \tag{2.1}$$

where $z = \eta(r)$ is the nondimensional elevation of the free surface, subject to

$$\frac{F^2}{2}(u^2 + v^2) + \eta - \beta \frac{(r\eta'' + \eta'(1 + \eta'^2))}{r[1 + \eta'^2]^{3/2}} = 0 \quad \text{on } z = \eta(r), \tag{2.2}$$

where the last term on the left-hand side represents the effect of surface tension, and

$$\phi_r \eta' - \phi_z = 0 \quad \text{on } z = \eta(r), \tag{2.3}$$

where the subscript denotes partial differentiation, is the usual kinematic condition that states that flow cannot be through the surface.

These equations include the main parameters that control this flow, the Froude number, F, and the nondimensional surface tension, β, given by

$$F = \left(\frac{m^2}{gH^5}\right)^{1/2}, \quad \beta = \frac{T}{\rho g H^2}, \tag{2.4}$$

in which ρ is the fluid density and g is the gravitational acceleration. In most cases, the Froude number can be thought of as an effective flow rate; large F values corresponding to strong flow.

In the limit as the point sink (with unit strength) is approached at $(r, z) = (0, -1)$, the velocity potential should take the form

$$\phi \to \frac{1}{4\pi \sqrt{r^2 + (z + 1)^2}}. \tag{2.5}$$

A change of sign reverses the flow direction from a sink flow to a source flow. However, in the case of steady flow, the quadratic nature of the velocity term in the dynamic condition (2.2) means that solutions generated apply for both source and sink flows.

3. Asymptotic solution

It is of interest to derive the solution for small Froude number and surface tension, both to analyse the flow and also for verification of the numerical solutions computed in the next section. Assuming that the Froude number is small, and hence that the disturbance to the free surface is small, we can linearize about a flat surface. Consider the expansions in powers of the Froude number:

$$\phi(r, z) = \phi_0(r, z) + F^2\phi_1(r, z) + O(F^4),$$
$$\eta(r) = F^2 Z_1(r) + F^4 Z_2(r) + O(F^6).$$

Substituting these expansions into the conditions described above (equations (2.1)–(2.5)), at first order we find that the potential ϕ_0 satisfies Laplace's equation (2.1) and the normal-derivative condition $\phi_{0z} = 0$ on $z = 0$. Together with the condition (2.5), the solution for ϕ_0 is found to be

$$\phi_0(r, z) = \frac{1}{4\pi}\left[\frac{1}{\sqrt{r^2 + (z + 1)^2}} + \frac{1}{\sqrt{r^2 + (z - 1)^2}}\right],$$

where the second term represents an image sink above the free surface.

The approximation at $O(F^2)$ to equation (2.2) yields a second-order ordinary differential equation in $Z_1(r)$ of the form

$$\beta Z_1''(r) + \frac{\beta}{r} Z_1'(r) - Z_1(r) = \frac{1}{2}\phi_{0r}^2(r, 0) = \frac{r^2}{8\pi^2(r^2+1)^3} \tag{3.1}$$

with boundary conditions $Z_1'(0) = 0$ and $Z_1 \to 0$ as $r \to \infty$.

It is possible to solve this problem exactly using Hankel transforms. If we let

$$Z_1(r) = \int_0^\infty A(k) J_0(kr) k \, dk, \tag{3.2}$$

and noting that from Bessel's equation $J_0''(kr) + (1/kr)J_0'(kr) = -J_0(kr)$, then (3.1) transforms, after re-arrangement, to

$$-\int_0^\infty k(1+\beta k^2) A(k) J_0(kr)\, dk = \frac{r^2}{8\pi^2(r^2+1)^3},$$

which can be inverted (Hankel inverse transform) to give

$$A(k) = -\frac{1}{8\pi^2(1+\beta k^2)} \int_0^\infty \frac{r^3}{(r^2+1)^3} J_0(kr)\, dr.$$

After substitution into (3.2), this can be converted to the form

$$Z_1(r) = \frac{F^2}{64\pi^2} \int_0^\infty \frac{k^2}{\beta k^2 + 1} [kK_0(kr) - 2K_1(kr)] J_0(kr)\, dk,$$

where K_0 and K_1 are Bessel K functions of orders 0 and 1, respectively [1]. These integrals do not appear to be easily solved, but it is a simple matter to evaluate them using quadrature.

Figure 1 shows a comparison between the linear solution and the full numerical solution for Froude numbers $F = 1.5, 3$ and 4.5 and surface tension $\beta = 0.02$. The linear solution compares well for even quite large values of $F = 3$ and 4.5. In the case $F = 1.5$, there is only a slight discrepancy at the bottom of the dip around the central stagnation point. As F increases, the comparison is not so good, as we would expect, with differences showing at the point of highest curvature near the bottom of the dip at around $r \approx 0.7$. The nonlinear effect is to pull the surface downward more, as it did in the analogous case of a line sink [4, 9]. It is this effect which is later shown to cause the breakdown of the solutions when surface tension is included.

4. The numerical method

To consider the full nonlinear steady flow problem we need to implement a numerical scheme using an approach similar to that of [3] and [12]. The flow is assumed to be axisymmetric and an integral equation is derived for the elevation and velocity potential on the free surface.

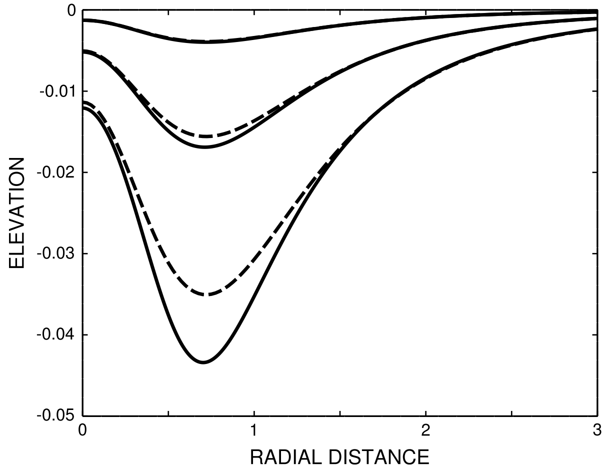

FIGURE 1. Comparison between the linear and the full numerical solutions for the surface shape for Froude numbers $F = 1.5, 3$ and 4.5 (curves top to bottom in order), with surface tension $\beta = 0.02$. The solid lines are the full nonlinear solution in each case and the dashed are the linear. The limiting solution for this case is at $F \approx 5.5$.

4.1. Formulation The formulation of the integral equation follows that given by Forbes and others [3, 11, 12]. The numerical scheme is described in detail by Hocking et al. [11]. For convenience, we briefly outline the derivation here.

We use Green's second identity to derive an integral equation for the unknown analytic function $\Phi(r, z)$ and surface elevation $z = \eta(r)$. Let Q be a fixed point on the free surface with coordinates $(r, \theta, \eta(r))$ and $P(\rho, \xi, \zeta)$ be another point which is free to move over the same surface. Since Φ is an analytic function over the full region except at the sink itself, we can define another function $\Psi = 1/R_{PQ}$ which is also analytic, except when P and Q are the same point, that is,

$$\Psi = \frac{1}{R_{PQ}} = \frac{1}{[r^2 + \rho^2 - 2r\rho\cos(\xi - \theta) + (z - \zeta)^2]^{1/2}}.$$

Invoking Green's second identity and noting that both Φ and Ψ satisfy Laplace's equation throughout the region enclosed by ∂V,

$$\iint_{\partial V} \left[\Phi \frac{\partial \Psi}{\partial n} - \Psi \frac{\partial \Phi}{\partial n} \right] dS = 0,$$

where n denotes the outward normal direction, and ∂V consists of the surface of the free surface S_T with the point Q carefully excluded by a small hemispherical surface, S_Q, and a small sphere about the sink, S_ϵ.

It is not difficult to show that the contributions from all of these surfaces leads to an integral equation of the form

$$2\pi\Phi(Q) = \frac{1}{(r^2 + (z+1)^2)^{1/2}} - \iint_{S_T} \Phi(P) \frac{\partial}{\partial n}\left(\frac{1}{R_{PQ}}\right) dS_P.$$

Following the work of Forbes and Hocking [3], the surface integral can be specified in terms of the variables of the problem as

$$2\pi\Phi(Q) = \frac{1}{(r^2 + (z+1)^2)^{1/2}} - \int_0^\infty \Phi(P)\mathcal{K}(a,b,c,d)\, d\rho,$$

in which the kernel function is

$$\mathcal{K}(a,b,c,d) = \rho \int_0^{2\pi} \frac{a - b\cos(\xi - \theta)}{[c - d\cos(\xi - \theta)]^{3/2}}\, d\xi$$

and the intermediate quantities a to d are defined as

$$a = \rho\eta_\rho(P) - (\eta(P) - \eta(Q)), \quad b = r\eta_\rho(P),$$
$$c = \rho^2 + r^2 + (\eta(P) - \eta(Q))^2, \quad d = 2r\rho.$$

Forbes and Hocking [3] reduced this to the form

$$\mathcal{K}(a,b,c,d) = \frac{4\rho}{d\sqrt{c+d}}\left[b\mathbf{K}\left(\frac{2d}{c+d}\right) + \left(\frac{ad - bc}{c - d}\right)\mathbf{E}\left(\frac{2d}{c+d}\right)\right],$$

where **K** and **E** are the complete elliptic integrals of the first and second kinds as defined by Abramowitz and Stegun [1]. At this point we note that **E** is well behaved over the interval of interest, but that **K** has a logarithmic singularity as $P \to Q$ in the integral over the free surface.

This problem was solved using a formulation based on arclength along the surface, so that s is the distance from $\rho = 0$ to Q, and σ is the distance along the surface to P. The standard formula

$$\left(\frac{dr}{ds}\right)^2 + \left(\frac{d\eta}{ds}\right)^2 = 1 \tag{4.1}$$

defines the arclength s in terms of r and η. We define a surface potential $\phi(s)$ and, applying the chain rule, we find that, along the surface,

$$\frac{\partial\phi}{\partial r} = \Phi_r(r,\eta) + \Phi_z(r,\eta)\frac{d\eta}{dr}.$$

Eliminating Φ_z from the Bernoulli equation (2.2) and the kinematic condition (2.3) and combining leads to a single relation

$$\frac{1}{2}F^2\left(\frac{d\phi}{ds}\right)^2 + \eta(s) - \beta\left[\frac{\eta''(s)}{r'(s)} + \frac{\eta'(s)}{r(s)}\right] = 0 \quad \text{on } z = \eta(r). \tag{4.2}$$

Rewriting the integral equation in terms of arclength,

$$2\pi\phi(s) = \frac{1}{(r^2(s) + (\eta(s) + 1)^2)^{1/2}} - \int_0^\infty \phi(\sigma)\mathcal{K}(A, B, C, D)\, d\sigma, \tag{4.3}$$

where

$$A = r(\sigma)\eta'(\sigma) - r'(\sigma)(\eta(\sigma) - \eta(s)), \quad B = r(s)\eta'(\sigma),$$
$$C = r^2(\sigma) + r^2(s) + (\eta(\sigma) - \eta(s))^2, \quad D = 2r(s)r(\sigma).$$

Forbes and Hocking [3] showed that this integral equation could be replaced by the nonsingular form

$$2\pi\phi(s) = \frac{1}{(r^2(s) + (\eta(s) + 1)^2)^{1/2}} - \int_0^\infty (\phi(\sigma) - \phi(s))\mathcal{K}(A, B, C, D)\, d\sigma, \tag{4.4}$$

as the extra term can be shown to be zero. This version allows an accurate quadrature scheme to be used (with care as $\sigma \to s$), and in this work cubic spline integration was used for all calculations. It is the results of this form (4.4) that are described in this paper rather than the product integration form described by Hocking et al. [11], although the results obtained using that method are almost identical to those presented here.

This integral equation is coupled with the condition (4.1), subject to (4.2), to give the complete formulation of the problem. The arclength formulation allows the method to find multiple-valued or overhanging free-surface shapes if they exist.

4.2. Computational details The equations derived in the previous section are highly nonlinear because of the quadratic dependence on velocity, the nonlinear surface-tension term and the fact that the surface shape is unknown. The equations were, therefore, solved numerically using collocation. A grid of points was chosen at values of arclength $s = s_0, s_1, s_2, s_3, \dots, s_N$. An initial guess for the surface shape $\eta = \eta_0, \eta_1, \eta_2, \dots, \eta_N$ and potential function $\phi = \phi_0, \phi_1, \dots, \phi_N$ on the surface was made and used to compute the error in the integral equation (4.3) and the condition on the surface (4.2). The initial guess was then updated using a damped Newton's method until the error in all equations dropped below 10^{-8}.

The numerical integration was performed using the method described by Forbes and Hocking [3], but with an algebraically increasing grid spacing. In a typical simulation, 1000 evenly spaced points were used in the interval $s \in [0, 5]$, and then another 1000 were spaced with a slowly increasing spacing until $s_N \approx 250$. As explained by Hocking et al. [11], the value of s_N needs to be very large to obtain convergence as the computational window is increased. At very small values of surface tension β, problems of convergence similar to those discussed in that paper [11] needed to be overcome, but for even moderate values of surface tension the convergence behaviour improved markedly. A number of simulations were performed to verify the results using different computational windows and grid spacings. The values suggested above as typical were found to produce consistent results.

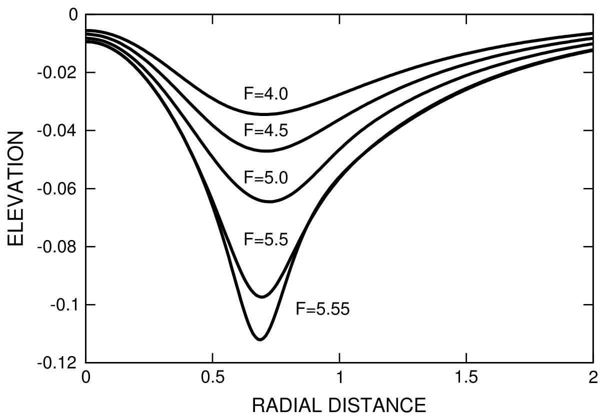

FIGURE 2. Free-surface shapes for $\beta = 0.01$ with $s_N = 10$ and different values of Froude number. The last case, $F = 5.55$, is the highest value for which steady solutions were obtained. It appears that the surface is deepening quickly at around $r = 0.7$. These shapes are typical of all of the moderate surface tension solutions.

5. Results

As indicated above, the solutions for the case where surface tension approaches zero are identical to those obtained by Hocking et al. [11]. The limiting steady value of $F \approx 3$ is significantly lower than those obtained by Forbes and Hocking [3] ($F \approx 6.4$) and Vanden Broeck and Keller [26] ($F \approx 5.4$). Computations with small values of surface tension were found to behave similarly. For example, using $\beta < 0.001$ gave solutions very similar to those with zero surface tension, and the limiting value of Froude number, while slightly higher, again seems to occur at the first sign of the formation of small wavelets on the surface. The inclusion of surface tension causes the central point above the sink to move downward slightly increasingly as the surface tension increases. In all cases, there is a dip that rings the central point at a distance of $r \approx 0.7$. The horizontal location of this dip is almost unaffected by different Froude numbers and surface-tension values. However, the depth of the dip is lower for higher surface-tension values at a fixed Froude number due to the stronger attractive force at the most curved parts of the surface, as can be seen in the surface condition (4.2).

Figure 2 shows the shape of the free surface as the Froude number is increased for $\beta = 0.01$. Importantly, the circular dip at around $r \approx 0.7$ becomes deeper at an increasingly rapid "rate" as the Froude number increases. The limiting solution seems to correspond to the formation of a much stronger dip. Figure 3 shows the depth of the bottom of the dip as a function of Froude number at several different values of surface tension β. The behaviour of the curves suggests that for moderate surface tension the change in dip depth with Froude number becomes vertical. The point at which these curves become almost vertical corresponds to the limiting Froude number. Figure 4 shows the dependence of the maximum Froude number on surface tension. The maximum Froude number increases as the surface tension increases, as it did

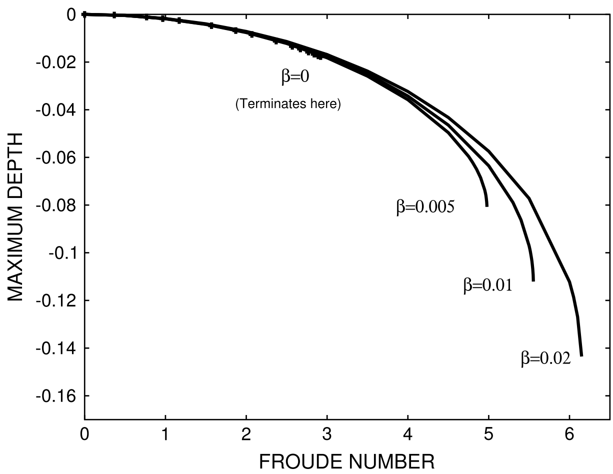

Figure 3. Plot of the deepest point on the free surface against Froude number for several different surface-tension values $\beta = 0, 0.005, 0.01$ and 0.02. The depth increases more quickly as F gets closer to the limiting steady value except in the case $\beta = 0$, where it appears to be the formation of small waves that limits the steady solutions.

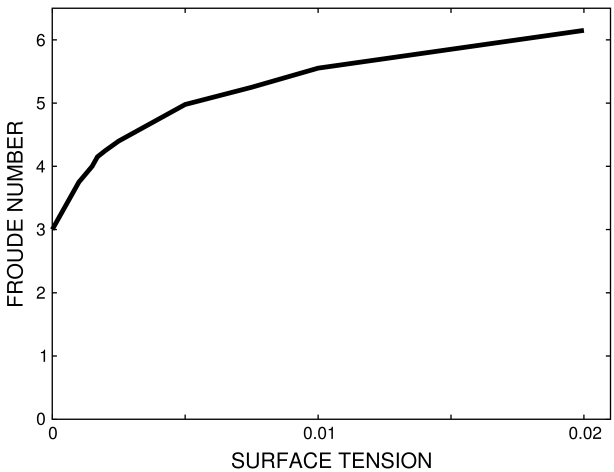

Figure 4. Plot of the maximum Froude number for each different value of surface tension. The value increases as the surface tension increases.

in the two-dimensional case, but here there was no evidence of the nonuniqueness found in that case [4, 9]. The maximum values of Froude number agree quite well with the values at which steady solutions could be obtained using full unsteady simulations [20].

6. Conclusions

We have considered the steady flow generated by a point sink or source submerged beneath a free surface. The numerical scheme has been verified by comparison with a linearized solution and then used to investigate the behaviour of the flow. The new results in this work are due to the inclusion of surface tension. At low values of surface tension, the behaviour was almost identical to the case with none. However, for moderate values around $\beta = 0.002$ and larger, it was found that the formation of a deepening trough appeared to be the cause of the cessation of the steady solutions. No evidence of a fold bifurcation such as that found in the case of flow into a line sink [4, 9] was found in this work.

The stability of these steady solutions has not been considered here, but it would certainly be of interest. The unsteady simulations presented by Stokes et al. [20] showed that the history of the flow was important in determining the outcome. A sudden initiation of the sink flow could cause an immediate and "catastrophic" drawdown of the free surface, if the flow rate was sufficient, but if the flow was increased slowly enough then this flow rate could be surpassed without drawdown, eventually evolving to a steady state like those computed here. The maximum values of Froude number obtained in that work, with the slowly evolving flow, agree quite well with the limiting solutions obtained in the current work.

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