A FAST & EFFECTIVE HEURISTIC FOR THE FEEDBACK ARC SET PROBLEM

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ABSTRACT

Let $G = (V, A)$ denote a simple connected directed graph, and let $n = |V|$, $m = |A|$, where $n - 1 \leq m \leq \binom{n}{2}$. A *feedback arc set* (*FAS*) of G, denoted $R(G)$, is a (possibly empty) set of arcs whose reversal makes G acyclic. A *minimum feedback arc set* of G, denoted $R^*(G)$, is a FAS of minimum cardinality $r^*(G)$; the computation of $R^*(G)$ is called the *FAS problem*. For every n, let $\rho(n)$ denote the maximum, over all digraphs G of order n, of $|R^*(G)|$. Berger and Shor have recently published an algorithm which, for a given digraph G, computes a FAS whose cardinality is $O(\rho(n))$. Thus the Berger/Shor algorithm provides, in a certain asymptotic sense, an optimal solution to the FAS problem. Unfortunately, the Berger/Shor algorithm is complicated and requires running time $O(mn)$. In this paper we present a simple FAS algorithm which guarantees a good (though not optimal) performance bound and executes in time $O(m)$. Further, for the sparse graphs which arise frequently in graph drawing and other applications, our algorithm achieves the same asymptotic performance bound that Berger/Shor does.

1 INTRODUCTION

The FAS problem defined in the Abstract was apparently first studied by Slater [S61] in the special case $m = \binom{n}{2}$ that G is a *tournament*. Interpreting the direction of each arc xy as a "win" for x over y in a round-robin tournament of n players, Slater looked for a minimum set of results (arcs) whose reversal would make the tournament consistent (acyclic). Much of the subsequent research was devoted to estimating $\rho_{_T}(n)$, the restriction of $\rho(n)$ to tournaments T. Then

$$\rho_{_T}(n) = \binom{n}{2}/2 - \phi_{_T}(n) \leq \rho(n),$$

where $\phi_T(n) \in O(n^2)$.

In 1965 Erdős and Moon [EM65] showed that

$$\phi_T(n) \in \Omega(n), \quad \phi_T(n) \in O(n^{3/2} log^{1/2} n).$$

The lower bound was sharpened to

$$\phi_T(n) \in \Omega(n log n)$$

by Jung [J71], and Spencer [S71,S80] was able to improve both bounds by showing that

$$\phi_T(n) \in \Theta(n^{3/2}).$$

Then the upper bound was further refined by de la Vega [d83], who proved that

$$prob[\phi_T(n) \leq 1.73 n^{3/2}] \longrightarrow 1$$

as $n \rightarrow \infty$, where all labelled tournaments on n vertices are regarded as equally likely.

Berger and Shor [BS90] provide a basis for both strengthening and generalizing these results. Given a digraph G of maximum degree Δ, their algorithm computes $R(G)$ of cardinality $r(G)$ satisfying

$$r^*(G) \leq r(G) \leq m/2 - c_1 m/\Delta^{1/2},$$

where c_1 is a positive constant. On the other hand, they exhibit an infinite class $\mathcal{G}$ of digraphs with the property that for every $G \in \mathcal{G}$,

$$r^*(G) \geq m/2 - c_2 m/\Delta^{1/2},$$

for some positive constant c_2. Since $\rho(n) \geq r^*(G)$, it follows that their algorithm computes $r(G) \in O(\rho(n))$. It follows also that, over all digraphs G such that $r^*(G) = \rho(n)$,

$$r^*(G) - m/2 \in \Theta(m/\Delta^{1/2}).$$

When G is a tournament ($m = \binom{n}{2}, \Delta = n-1$), this reduces to the form established by Spencer for $\rho_T(n)$.

Since the FAS problem is NP-hard [K72], it is likely that no polynomial-time algorithm computes $r^*(G)$ exactly; hence the asymptotic performance bound of the Berger/Shor algorithm can be regarded as best possible. At the same time, Berger/Shor requires $O(mn)$ execution time, and it may be that, in many circumstances (for example, for on-line processing), a much faster algorithm with a suboptimal but good performance bound would be preferred. Such an algorithm, with performance bound

$$r(G) \leq m/2 - n/6$$

and $O(m)$ execution time, is presented in Section 2 of this paper. Observe that, over digraphs satisfying $m \in O(n)$, this performance bound is at least as good

as that of Berger/Shor, and is in fact better over digraphs for which, in addition, $\Delta \notin O(1)$.

Section 3 briefly discusses other FAS algorithms and makes suggestions for future work.

2 A NEW FAS ALGORITHM

Suppose that the vertices of G are arranged in some order on a horizontal line and labelled $v_1, v_2, \ldots, v_n$ from left to right. We call such an arrangement a *vertex sequence* and denote it by $s = v_1 v_2 \ldots v_n$. Clearly each vertex sequence s induces a feedback arc set $R(s)$ consisting of all the leftward arcs $v_j v_i$, $j > i$; moreover, every feedback arc set corresponds to some vertex sequence. Thus the FAS problem is equivalent to the problem of determining a vertex sequence s^* such that $R(s^*) = R^*(G)$.

In this section we present a greedy algorithm, called Algorithm GR, which computes a "good" vertex sequence s (one corresponding to a "small" feedback arc set $R(s)$). The algorithm greedily removes from G vertices (and their incident arcs) which are sinks or sources, or which satisfy a property that we now specify. For any vertex $u \in V$, let $d(u)$ denote the degree of u, $d^+(u)$ its outdegree, and $d^-(u)$ its indegree: then $d(u) = d^+(u) + d^-(u)$. At each step of Algorithm GR, after sinks and sources have been removed, a single vertex u is removed for which $\delta(u) = d^+(u) - d^-(u)$ is currently a maximum. If a vertex u removed from G is a sink, it is concatenated with a vertex sequence s_2; otherwise, u is concatenated with a vertex sequence s_1. When G has been reduced to an empty graph by successive removals, the output vertex sequence s is computed by forming the concatenation $s = s_1 s_2$. A formal statement of the new algorithm is given below:

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procedure GR (G : DiGraph; var s : VertexSequence);
s1 ← ∅; s2 ← ∅;
while G ≠ ∅ do
    {while G contains a sink do
        {choose a sink u; s2 ← u s2; G ← G − u};
     while G contains a source do
        {choose a source u; s1 ← s1 u; G ← G − u};
     choose a vertex u for which δ(u) is a maximum;
     s1 ← s1 u; G ← G − u};
s ← s1 s2.
```

We consider first the performance of this algorithm.

Theorem 2.1 Algorithm GR computes either an empty vertex sequence or a vertex sequence s for which $|R(s)| \leq m/2 - n/6$.

Proof For $n \leq 1$, GR computes $s = \emptyset$. Otherwise, GR induces a partitioning $V = \bigcup_{i=1}^{5} V_i$ where

* V_1 consists of removed vertices u which at the time of removal were sinks of indegree $d^-(u) > 0$;

* V_2 consists of removed vertices u which were isolated ($d(u) = 0$);
* V_3 consists of removed vertices u which were sources of outdegree $d^+(u) > 0$;
* V_4 consists of removed vertices u such that $d^+(u) = d^-(u) > 0$;
* V_5 consists of removed vertices u such that $d^+(u) > d^-(u) > 0$.

For $1 \leq i \leq 5$, let $n_i = |V_i|$ and let m_i denote the number of arcs removed from G as a result of the removal of the vertices of V_i. Then $n = \sum_{i=1}^{5} n_i$, $m = \sum_{i=1}^{5} m_i$, and $m_2 = 0$.

Since G is connected and $n > 1$, there is initially no isolated vertex. Suppose that removal of a vertex $u \in V_4 \cup V_5$ induces an isolated vertex v. Then v must result from removal of either uv or vu, but not both. Hence v would have been either a sink before the removal of uv or a source before the removal of vu. In either case v would already have been removed, and so removal of u cannot give rise to a vertex $v \in V_2$. A similar argument shows that removal of $u \in V_3$ cannot induce an isolated vertex v, and we see then that v can only be induced by prior removal of a sink u and an incident arc vu. It follows that

$$n_2 \leq m_1. \tag{2.1}$$

Observe that if GR removes a vertex $u \in V_4$, the resulting digraph $G - u$ contains exactly $d^+(u)$ vertices v for which $d^+(v) = d^-(v) + 1$ and exactly $d^+(u)$ vertices w for which $d^-(w) = d^+(w) + 1$. Since as we have seen $G - u$ contains no isolated vertex, it follows that the next removed vertex $x \in V_1 \cup V_3 \cup V_5$. That is,

$$n_4 \leq n_1 + n_3 + n_5,$$

which, by substitution for n_4, becomes

$$\begin{aligned} n &\leq 2n_1 + n_2 + 2n_3 + 2n_5 \\ &\leq 2n_1 + n_2 + 3n_3 + 3n_5, \end{aligned}$$

so that, applying (2.1) and observing that $n_1 \leq m_1$, we find

$$n \leq 3(m_1 + n_3 + n_5). \tag{2.2}$$

Consider now the number of leftward arcs induced by the removal from G of a vertex u and its insertion in the vertex sequence s. If $u \in V_4$, then the number of leftward arcs to u at the time of its removal is exactly $d(u)/2$; if $u \in V_5$, then the number of leftward arcs to u at the time of its removal is at most $(d(u) - 1)/2$. A removed vertex $u \in V_1 \cup V_2 \cup V_3$ contributes no leftward arcs. Thus,

$$\begin{aligned} r(s) = |R(s)| &\leq m_4/2 + (m_5 - n_5)/2 \\ &= m/2 - (m_1 + m_3 + n_5)/2 \\ &\leq m/2 - (m_1 + n_3 + n_5)/2, \end{aligned}$$

since $n_3 \leq m_3$. Applying (2.2), we obtain the desired result. $\square$

The upper bound of Theorem 2.1 is sharp, since it is attained when G is a directed cycle of length 3. However, for tournaments, a slightly sharper bound can be stated, as follows:

Theorem 2.2 Given a tournament G, Algorithm GR computes a vertex sequence s for which $|R(s)| \leq m/2 - \lfloor n/2 \rfloor /2$.

Proof Suppose that n is odd and observe that removing the first vertex u from G contributes at most $(n-1)/2$ arc reversals to $r(s)$. $G-u$ will then be a tournament of even order $n-1$, and removing a vertex v from $G-u$ contributes at most $\lfloor (n-2)/2 \rfloor$ arc reversals. In general,

$$\begin{aligned} r(s) &\leq (n-1)/2 + \lfloor (n-2)/2 \rfloor + \cdots + 4/2 + \lfloor 3/2 \rfloor + 2/2 \\ &= (n-1)^2/4. \end{aligned}$$

A similar calculation yields

$$r(s) \leq n(n-2)/4$$

when n is even. Both these inequalities reduce to the required form. $\square$

We now consider the implementation of Algorithm GR. For this, it is convenient to partition the vertex set of G into sources, sinks, and δ-classes, as follows:

$$V_d = \{u \in V \mid d = \delta(u); d^+(u) > 0; d^-(u) > 0\}, -n+3 \leq d \leq n-3;$$

$$V_{-n+2} = \{u \in V \mid d^+(u) = 0\};$$

$$V_{n-2} = \{u \in V \mid d^-(u) = 0; d^+(u) > 0\}.$$

It is clear that every vertex $u \in V$ falls into exactly one of these $2n-3$ classes. To compute these classes for a given digraph G, Algorithm GR needs to perform a bin sort as an initialization phase. The bin sort implements each vertex class as a bin, with the vertices in each class linked together by a doubly linked list. This structure can be formed in $O(m)$ time and uses $O(m)$ space.

Using the bins, it becomes trivial to recognize a sink u, a source u, or a vertex u for which $\delta(u)$ is a maximum. Furthermore, having determined u, it becomes trivial to compute s_1 or s_2 and, at the same time, to output the leftward arcs, if any, currently incident at u.

It remains only to be seen how to form $G-u$. The vertex u itself can be removed in $O(1)$ time by eliminating it from its bin list. As a result, every vertex v adjacent to u will either become a sink, a source, or an element of an adjacent bin: to perform these updates requires $O(1)$ time for each arc incident at u. We have proved

Theorem 2.3 Algorithm GR can be implemented in $O(m)$ time and $O(m)$ space.

3 REMARKS

In Section 2 we described Algorithm GR, a simple new FAS algorithm with a good (though not optimal) performance bound and $O(m)$ (linear) execution time. For sparse digraphs this performance bound is at least as good as that of Berger and Shor, whose algorithm requires execution time $O(mn)$. However, for dense digraphs, the Berger/Shor algorithm has a performance bound which is, in a certain sense, optimal.

Our new algorithm is the only known FAS algorithm which executes in linear time and has a performance bound less than $m/2$ for all digraphs. Trivial linear algorithms [ELS89,L92] have been known for some time, but these do not even guarantee good performance bounds for tournaments. [ELS89] describes new greedy and divide-and-conquer algorithms which execute in linear time and guarantee good performance bounds for tournaments; but, as shown in [L92], there nevertheless exist digraphs for which these algorithms yield feedback arc sets of cardinality greater than $m/2$. ([L92] also describes an $O(mn)$ FAS algorithm (a greedier version of Algorithm GR) whose performance bound is not as good as Berger/Shor in general, but which nevertheless yields slightly better results when applied to oriented cubic graphs.)

In a practical context, the real challenge is to find a technique for expressing performance bounds, not just in terms of $\rho(n)$, but rather in terms of $r^*(G)$. For many digraphs G, $r^*(G)$ is much smaller than $\rho(n)$, and in such cases there is persuasive computational evidence [ELS89] that even simple linear algorithms will "usually" yield feedback arc sets of cardinality not much greater than $r^*(G)$. Thus, to find a means of expressing performance bounds of certain algorithms (such as Algorithm GR and the Berger/Shor algorithm) in terms of $r^*(G)$ would provide a much sounder basis for evaluating their practical effectiveness.

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