PROBLEMS & CONJECTURES ON SUM GRAPHS

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EXTENDED ABSTRACT

Given an integer $r \geq 0$, let $G_r = (V_r, E)$ denote a graph consisting of a simple finite undirected graph $G = (V, E)$ of order n and size m together with r isolated vertices $\overline{K}_r$. Then $|V| = n$, $|V_r| = n + r$, and $|E| = m$. Let $L : V_r \to \mathbb{Z}^+$ denote a labelling of the vertices of G_r with distinct positive integers. Then G_r is said to be a *sum graph* if there exists a labelling L such that for every distinct vertex pair u and v of V_r, $(u, v) \in E$ if and only if there exists a vertex $w \in V_r$ whose label $L(w) = L(u) + L(v)$. For a given graph G, the *sum number* $\sigma = \sigma(G)$ is defined to be the least value of r for which G_r is a sum graph.

Sum graphs were introduced by Harary [Har88]. In 1989 Gould and Rödl [GR89], using non-constructive methods, showed that there exist infinite classes $\mathcal{G}$ of graphs such that, over $G \in \mathcal{G}$, $\sigma(G) \in \Theta(n^2)$. In the same year Hao [Hao89] established a lower bound on $\sigma(G)$ in terms of the degree sequence of G, and also showed that a sum graph of order n and size m exists if and only if

$$m \leq \frac{\binom{n}{2} - \lfloor \frac{n}{2} \rfloor}{2}.$$

These results have recently been applied by Smyth [S92] to *unit graphs* (that is, graphs G for which $\sigma(G) = 1$). He shows that there exists a unit graph of order $n > 1$ and size m if and only if $\lceil n/2 \rceil \leq m \leq \lfloor n^2/4 \rfloor$, and provides a methodology for constructing at least one such graph for each suitable value of m. The same paper also shows how to construct graphs G of given order $n \geq 4$ and size m whose sum number $\sigma(G) \leq 2n - 3$. Other constructions have been found for specific classes:

* Ellingham [E89] shows that every nontrivial tree is a unit graph;
* Bergstrand et al. [BHHJKW89] show that for a complete graph K_n, $n \geq 4$, $\sigma(K_n) = 2n - 3$;
* Hartsfield and Smyth [HS92a] show that for a complete bipartite graph $K_{p,q}$, $2 \leq p \leq q$, $\sigma(K_{p,q}) = \lceil (3p + q - 3)/2 \rceil$;
* Hartsfield and Smyth [HS92b] show that for wheels W_n of order $n + 1$ and size $m = 2n$, $\sigma(W_n) \in \Theta(m)$ (for all other classes $\mathcal{G}$ of graph whose sum number is known, $G \in \mathcal{G}$ implies that $\sigma(G) \in o(m)$).

Thus to date constructions have mainly focussed on sum graphs of "small" sum number: no class of graphs is known whose sum number is even close to the bound of Gould and Rödl. In this paper we present a collection of problems and conjectures arising out of the work referenced above.

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