# Conceptual development and the modern scientific calculator: Using a forgotten technology

*Barry Kissane & Marian Kemp*
b.kissane@murdoch.edu.au m.kemp@murdoch.edu.au
Murdoch University
AUSTRALIA.

**Abstract**: *Calculators have frequently been regarded only as devices to perform calculations and thus often regarded with disapproval by mathematics teachers. With the availability of sophisticated technologies in many settings, it seems that the potential for scientific calculators has been neglected recently, and developments in this technology not adequately exploited. This is of particular significance in developing countries, where resources are limited. In this analytic paper, we highlight some opportunities created for conceptual development with regular access to a modern scientific calculator. The focus is on the development and deep understanding of mathematical concepts, widely recognized as of prime importance to student learning. The analysis is illustrated with examples related to the multiple representation of concepts and to the use of an advanced scientific calculator to provide numerical experience of important mathematical concepts.*

## 1. Introduction

Early versions of calculators around 50 years ago were restricted in operations to elementary arithmetic so that it is unsurprising that they were regarded as essentially tools to undertake numerical computation. Similarly, when the first hand-held scientific calculators were developed some 40 years ago, they extended the range of computations directly accessible (for example, by including logarithmic and exponential functions), so that their designation as 'calculators' is also unsurprising. The adjective, 'scientific', referred to the provision of capabilities to be used by scientists (and engineers), essentially for the purpose of completing calculations efficiently.

Since that time, calculators have developed substantially and have become widely used by people other than scientists and engineers, but there continues to be a pervasive view that the essential purpose of the devices has not changed. The principle aim of this paper is to challenge that conception of calculators – and scientific calculators in particular – and to recognize their primary roles in representing many different aspects of mathematics and hence supporting students to learn about these. Over the past 40 years, the scientific calculator has been developed so that it is no longer principally a tool for scientific computation, much of which would these days be done on computers, but instead is a tool for mathematics teachers and their students: an educational tool.

As a consequence of the process of developing and improving technology for mathematics education, rather than only for calculation, there is now a wide range of mathematical concepts that can be developed or enhanced through the use of modern calculators (not only sophisticated calculators), particularly advanced scientific calculators. They include the concepts of fractions; decimals; the solution of equations; functions; numerical differentiation and integration; matrices; vectors; complex numbers; recursion; elementary statistics and probability. The calculator capabilities to develop and access these concepts have been designed and moulded by calculator manufacturers to support education, rather than the computational needs of scientists and engineers – the original users of handheld calculators around four decades years ago.

In addition to widening the range of mathematical ideas that are treated by the calculators, substantial developments in calculator interfaces and screen representations have taken place. While early calculators used LEDs to show numbers, more recent versions show mathematical symbols and notations, in a form increasingly similar to standard textual representations. Thus, on modern calculators, fractions are represented as fractions (with a horizontal vinculum), roots are shown with radical signs, integrals are shown with integral signs, powers are shown as raised numbers, and so on. These enhancements have increased the extent to which the modern hand-held scientific calculator can be reasonably regarded as developed to be an everyday companion device for students studying mathematics in school, rather than an object developed for a different environment and set of purposes.

Despite these developments, there continues to be a level of unease in many countries about the place of calculators in schools, and reluctance locally and officially to allow the devices to be used in exams or supported in the classroom. Consequently, a major purpose of this paper is to outline and illustrate the case for the use of calculators to enhance mathematics learning in schools and the early years of universities, in contrast to expressed views that the use of calculators is likely to inhibit learning. In the 1970's, when scientific calculators were first available, [2] describe the educational debate regarding their appropriateness to mathematics education; it seems that some parts of this debate remain unresolved, although modern scientific calculators have changed remarkably since that time.

In a previous paper [5], we have described in general terms the effects on curriculum development of making various kinds of decisions regarding calculator use. This paper describes in some detail some of the ways in which scientific calculators in particular can be used to enhance and enrich school mathematics, supporting rather than undermining its intentions. Contrary to some common views, the paper argues that a major role for the calculator is to help create meanings for mathematical ideas, rather than merely to permit users to obtain (or to check) numerical answers to computational questions.

Of particular importance to the relationship of scientific calculators to mathematics education is the role of technology in developing countries, where there are relatively limited resources available to individuals for school. We have argued elsewhere [4] that a handheld calculator, especially a modern advanced scientific calculator, which performs most of the functions of a graphics calculator but without a graphics screen, continues to represent the best technology investment for schools and individuals in these countries. Such a technology is the most likely to be available on a wide enough scale to allow curricula to develop accordingly across a country.

## 2. Research literature

Somewhat surprisingly, there does not seem to be an extensive research literature on the use of *scientific* calculators for learning mathematical concepts. Early research work up to the middle of the 1980s on calculators seemed concerned with understanding their impacts on the development of mathematical skills, on students using the calculators as tools to solve problems and on the subsequent effects of calculators on student attitudes [10]. After that time, interest was focused on the use of graphics calculators, which began to be used widely in some settings, followed closely by algebraic calculators, which included computer algebra systems. In addition the decade between 1985 and 1995 saw the introduction of the Internet and sophisticated software for mathematics (notably dynamic geometry systems), so it may be that the scientific calculator became a forgotten tool, especially in relatively affluent countries where attention of educators and researchers shifted to newer technologies of potential for mathematics education. Indeed, a typical modern view about scientific calculators might be that expressed by [3], emphasizing their capacity to save time on calculations:

> … calculators would allow the students, who could perform these manipulations and computations quickly and accurately, think and reason mathematically. They do not have to waste unnecessary time on wrong computation and teachers and students would be able to interact more effectively using mathematical vocabulary to understand the concept better. (p. 160)

Using meta-analyses, [10] synthesized the available research on the use of calculators in mathematics education, including a range of kinds of calculators in their meta-analyses. They noted the consistency of positive findings in research studies, and noted (with a slight air of frustration):

> Few areas in mathematics education technology have had such focused attention with such consistent results, yet the issue of whether the use of calculators is a positive addition to the mathematics classroom is still questioned in many areas of the mathematics community, as evidenced by continually repeated studies of the same topic. As a result, we concluded that future practitioner questions about calculator use for mathematics teaching and learning should advance from questions of whether or not they are effective to questions of what effective practices with calculators entail. (p. 2)

When scientific calculators began to be introduced into classrooms early in the 1970s it was expected that they would allow for: extended experimental and discovery learning; concept formation on a broad numerical basis; and more meaningful algorithmic calculation [2, p. 665]. These expectations still seem reasonable today and arguments for the use of scientific calculators would include all three of these reasons. More advanced scientific calculators available today are much more sophisticated than they were in the 1970s, however, and permit students to develop more complex concepts in more than just numerical fields as they can study matrices, sequences and series, integrals and derivatives, vectors and more, developing concepts alongside developing expertise in using the scientific calculator.

There appears to be very little empirical evidence concerning the use of the scientific calculator, further than to endorse its use in general for learning mathematics. This paper does not present empirical evidence for the use of scientific calculators; indeed, in the western world the majority of students in senior secondary schools now use graphics calculators or CAS calculators, and scientific calculators have not been carefully studied in recent years, despite being almost universally available. In developing countries such as Kenya [8] and Zimbabwe [7], however, the situation is different and there are major challenges to the use of scientific calculators. These authors cite poverty, parental ignorance regarding the need to have calculators and feelings that the government should provide calculators as reasons for students to not have access to calculators. Some traditional educators in Zimbabwe were influential in curriculum development, so that two parallel curricula were developed in the 1990s, one using logarithms and the other using scientific calculators.

However, even when students do have access to calculators, this does not ensure that they are used to develop mathematical concepts, if teachers use them in a conservative way, focusing on answering numerical questions. For example, the survey of the field in [11] emphasised the important role of teachers, as not only is their mastery of calculator skills recognized as essential, but their personal philosophies and dispositions towards integration of calculators into mathematics learning are seen to be vital. Similarly, [1] remind us of the importance of teachers' beliefs on their pedagogies. They note that teachers who believe that scientific calculators are used merely to replace students' need to learn basic facts and computational skills will merely use the calculators as computational tools. However, other teachers who believe that students can learn mathematical concepts and critical skills through calculator use will be more likely to integrate their use into the classroom.

Issues of these kinds are evident in the developing world. For example, a survey of Zimbabwean teachers in relation to O-Level curricula with and without scientific calculators concluded that teachers' responses did not indicate a sound understanding of pedagogical purposes for using calculators, despite being confident to use them [7]. Thus, the survey found that only very few teachers used calculators to introduce or to develop concepts, although almost half of the teachers reported using them to facilitate explorations. Similarly, a report on student experiences of learning mathematics with scientific calculators in Kenya [8] observed that 66% of respondents were not able to use the calculators effectively, apparently due to the lack of adequate preparation of teachers and limited access to calculators. They noted that teachers did not typically teach students how to use the calculators in advance of learning about mathematical concepts.

Challenges and issues of these kinds are part of many classrooms throughout developing regions in Asia and Africa, in which the lack of financial resources seems inevitably to play a negative part in the realization of mathematical learning with scientific calculators.

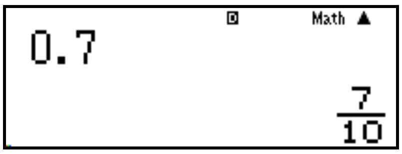
## 3. Examples

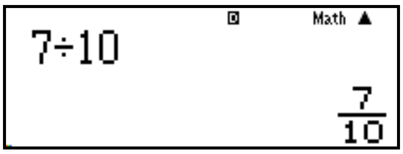
A modern advanced scientific calculator can provide opportunities for students to develop mathematical concepts in several different ways. In this section, we describe and illustrate some of these ways, taken from our recent developmental work [6]. A key feature of this work has concerned using the calculator to create meanings for mathematical ideas, rather than to merely undertake computations. Constrained by the length of the paper, we illustrate opportunities of two different kinds: (i) those using the calculator capacity to represent concepts in several different ways and (ii) those associated with numerical representation of concepts. The illustrative calculator screen dumps that follow are all produced with a CASIO fx-991 ES PLUS advanced scientific calculator.

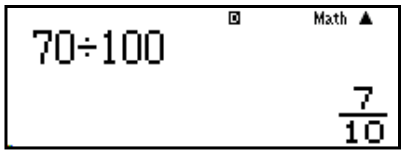
### 3.1 Multiple representations

The capabilities of calculators to represent mathematical objects in different ways provides the potential for students to appreciate the significance and the meaning of those different representations. While this has been frequently referred to in discussion about graphics calculators, modern advanced scientific calculators have many of the same features of graphics calculators (with the exception of graphing capabilities), leading to multiple representations of various kinds.

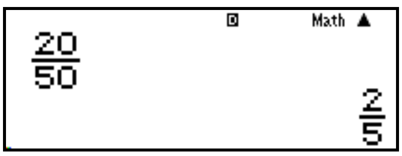
A good example of this at an elementary level concerns fractions and decimals. While these two concepts have often been regarded as separate in the mathematics curriculum and thus unrelated, or at least only slightly related, a modern calculator emphasises that these concepts can be regarded as merely different ways of representing the same number. In this regard, the calculator screens below show three different examples of how decimals and divisions are routinely represented by fractions. (In each case, the first line of the display shows what a user has entered into the calculator, while the second line of the display shows the calculator's response.)

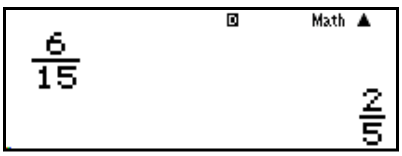


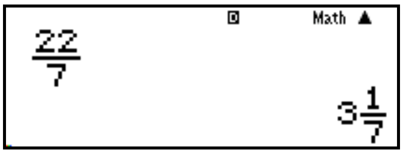




In a similar way, many calculators routinely represent fractions in their least complicated form, so that, rather than being regarded as a school exercise of 'simplifying' or 'cancelling', equivalent fractions can be understood as different ways of representing the same numbers, as the examples below indicate:

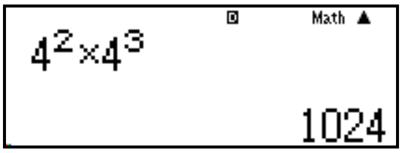


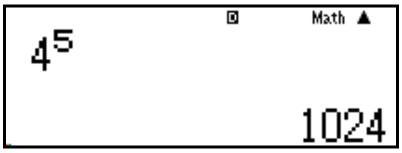



Rather than being regarded as separate ideas to be learned, with technologies for moving between them, proper and improper fractions can also be understood as merely different ways of representing a fraction larger than unity:

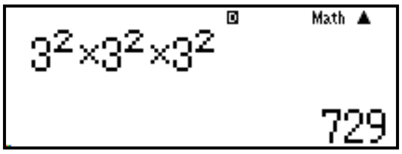



The appearance of a fraction to decimal key on many scientific calculators has allowed students at will to move between these two different representations of a number. Powers of numbers, and laws of indices can be represented on modern advanced scientific calculators to expose and exemplify important relationships:



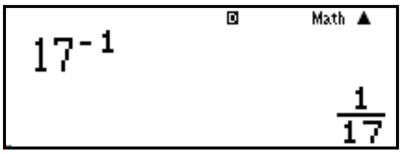



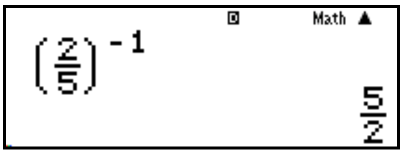

Calculator operations such as those below need to be understood as focusing attention on the important relationships involved between indices, rather than as being mostly of value for determining the answer to a numerical question:

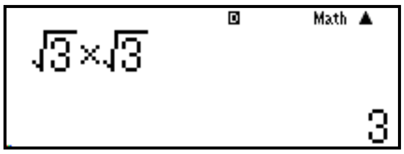




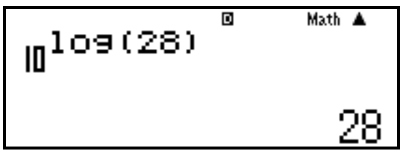
Negative powers of numbers are routinely represented in revealing ways on a modern calculator, which can be helpful to students learning about their meanings, as the first two screens below show. The essential meaning of square roots is illustrated by relationships such as that shown in the third screen below:

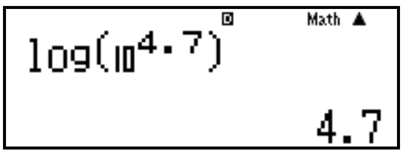


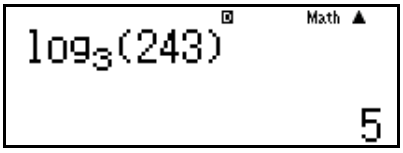



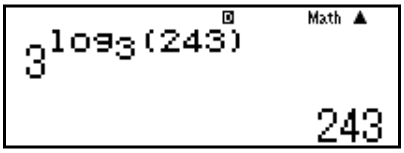

As a final example, and with more sophisticated students in mind, the idea of a logarithm is seen to be intrinsically related to powers of as number (the base), when students are given an opportunity to use a scientific calculator to represent these in various ways.



The calculator naturally allows a student to see that 10 raised to the logarithm (to base 10) of a number necessarily produces the number concerned, and permits students to experiment with this to see that it seems to hold in general. This is a crucial aspect of the concept of a logarithm, although it is frequently difficult for students to grasp. In a related way, a modern calculator allows students to see that the concept of a logarithm is a quite general one, not restricted to a base of 10 (or of *e*), as the following screens suggest:




Overall, our work [6] shows many examples of multiple representations, where the capability of the scientific calculator to represent mathematical objects in different ways provides an opportunity for students to grasp the essential ideas and relationships in new ways, not as readily available through the use of paper and pencil.

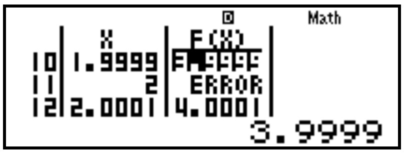
**3.2 Numerical representation of key ideas**

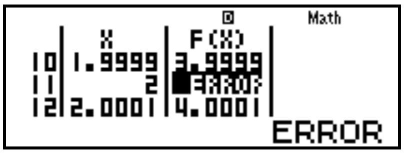
Some advanced mathematical ideas are recognized by teachers as difficult for students to grasp. A scientific calculator can contribute to students' conceptual development by providing a means of representing and exploring the ideas through numerical approximations.

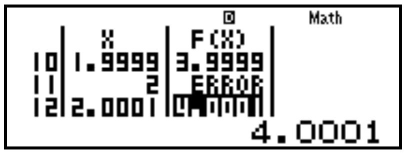
For example, the concept of a limit can be represented on a calculator by examining values that are successively closer to, but not quite reaching, a limiting value. To explore the limit:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

a calculator can be used to tabulate values 'close' to $x = 2$, as shown below:



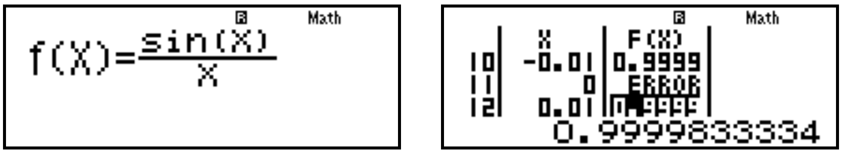




Rather than an abstract idea of 'closer and closer', students can see for themselves in a table what seems to happen as values for *x* approach (but do not reach) 2. The calculator lends itself to an iterative process, through appropriate use of a table in this case. In one sense, this is not a new idea, as mathematics teachers for generations have tried to help students understand the conceptually challenging idea of approaching a limit; what is new is the availability of a technology to explore this idea readily without needing to spend excessive time on undertaking the necessary computations.

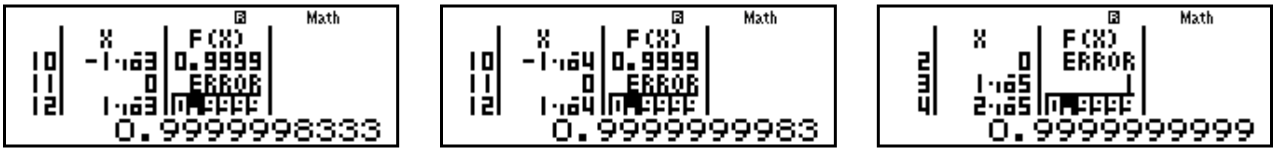
A standard treatment of introductory calculus for circular functions involves the key limit result:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Tables of values for *x* with successively closer to zero provide a representation of the concept of a limit. The screens above show that the function is not defined for $x = 0$, but seems to have a value close to 1 for values of *x* near zero.

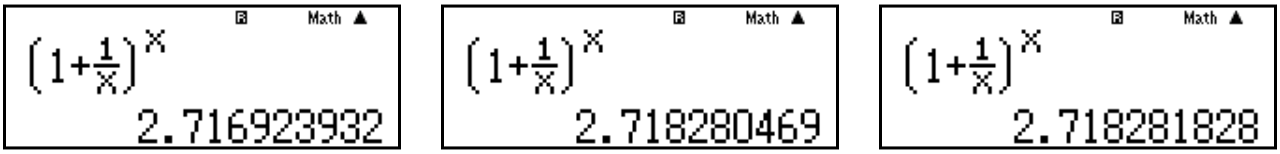



Increasingly smaller intervals show that, as *x* becomes extremely close to zero, the value of the function approaches 1. (In reading some screens below, notice that some *x*-values are shown in scientific notation, for space reasons. E.g., -0.001 is shown as -1 x $10^{-3}$ in the first screen.)




In the final screen above, the calculator displays a value of *f*(0.00001) = 1, as the best approximation to the actual value (which is not quite 1). Teachers should be careful here to make sure that students do not misinterpret this display, but to use it productively to understand the nature of both a function approaching a limit and the nature of representing numbers on screens.

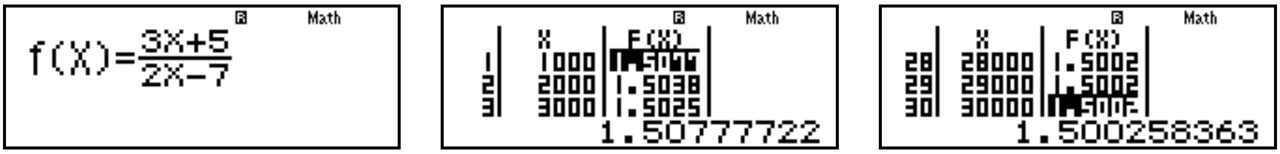
Tabulation is not the only mechanism that can be used to represent limits. Students can explore for themselves on a calculator the difficult concept of limits to infinity, through the mechanism of choosing ‘large’ values of the variable and noticing the consequences. A succession of choices (each with an increasingly large value of the variable) can show a process of converging to a limit. For example, the three screens below show the expression evaluated at *x* = 1000, *x* = 1 000 000 and *x* = 10 000 000 000:



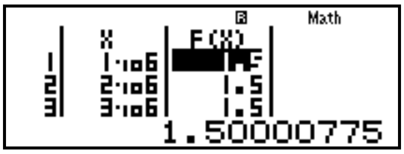

Of course, mathematical proofs are needed to establish limits, but the calculator can display good numerical approximations in these sorts of ways, and develop some in sight into the nature of the phenomenon involved. In this case, the screens above reflect the important mathematical result:

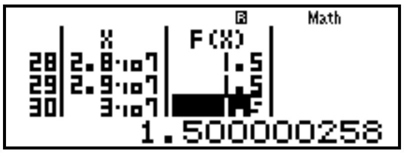
$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \approx 2.718281828$$

Asymptotic limits of these kinds can also be used to explore the concept of convergence to a limit at infinity, using tabulation.

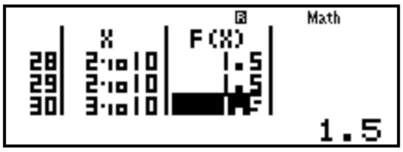



This kind of table suggests that the limit is close to 1.5. Choosing successively larger values for the table parameters suggests the same result, even more clearly, as shown below:






In the two screens above, the tabulated values are all shown as 1.5, because of the screen limitations, but the actual value is a little larger than this, as shown at the bottom of each screen. The screen below shows the result of using one billion for the Step in the table, and suggests even more strongly that the limit value is 1.5. In this case, the value at the bottom of the screen is also showing as 1.5, because of the screen resolution, but in reality it is a little more than this.

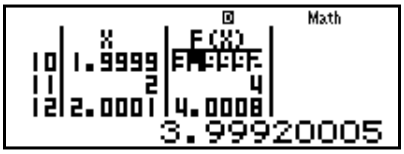


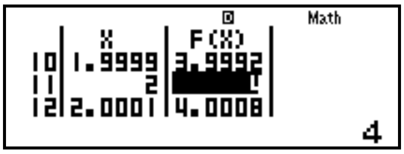

These calculator results are consistent with the following formal result that can be proved mathematically:

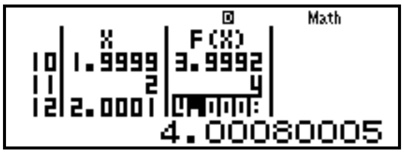
$$\lim_{x\to\infty}\frac{3x+5}{2x-7}=\frac{3}{2}.$$

The purpose of using the calculator is not as a substitute for the analytical ideas involved here, but to provide a concrete realization and approximation of the ideas, which are important for developing the concept of a limit.

In a similar way, concepts of continuity and discontinuity can be supported through appropriate use of a calculator. A defining feature of continuous functions is that small changes in a variable are associated with small changes in the function itself. On a calculator, this can be explored by tabulating the function appropriately, as shown below for the continuous function $f(x) = x^3 - x^2$ near $x = 2$.

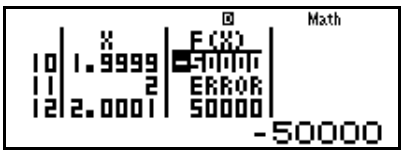


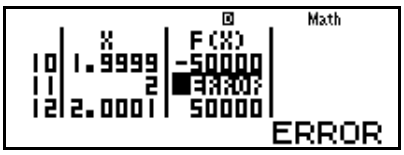


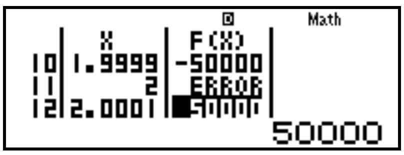



The screens suggest that the function is continuous at $x = 2$. Choosing smaller intervals for $x$ will still result in small changes for $f(x)$ when the function is continuous. When a function is discontinuous, however, values of the function change dramatically over some small intervals. The example below shows the jump discontinuity at $x = 2$ of the function

$$f(x)=\frac{5}{x-2}$$

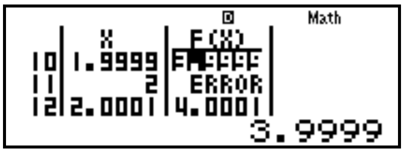


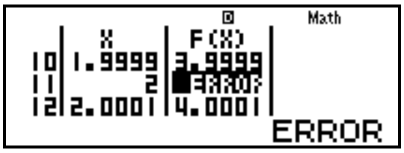


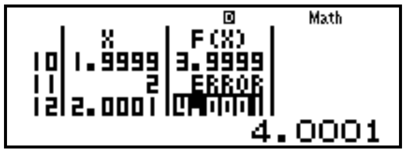


In this case, the values of the function jump from $f(1.999) = -50\ 000$ to $f(2.001) = 50\ 000$. In addition, the function is not defined for $x = 2$ (as shown by the error message). The table indicates that the function is discontinuous at $x = 2$ and provides insight into the nature of the discontinuity.

As a further example, consider the removable discontinuity at $x = 2$ of $f(x) = \dfrac{x^2 - 4}{x - 2}$ .




In this case, the function is not defined for $x = 2$, but the values of $f(x)$ do not jump on either side of $x = 2$. In tables for the function on even smaller intervals, the same phenomenon will occur. In fact, for all values *except* $x = 2$, the function can be expressed as $f(x) = x + 2$.

## 4. The role of the teacher

Our analysis suggests that calculators can be used productively to support the development of mathematical concepts. However it is important to note the critical role of the teacher in ensuring that this potential is realized, as reflected in some research reported earlier.

[9] were concerned with Mathematical Analysis Software (MAS) including a wide range of calculators from four-function, scientific, graphic or computers which may be used individually or shared on a screen for all students in a class. These are devices which they note can be used to "perform the algorithmic procedures from any branch of mathematics" (p.2) They noted the importance of the didactic contract, described as "about reciprocal responsibilities and expectations of the teacher and students with respect to mathematical knowledge" [9, p.2] Under such a contract teachers expect students to learn in a way that they establish such as traditional or non-routine, and students have a responsibility to learn.

From this perspective, it is clear that the teacher has a major role in directing, orchestrating and managing how the calculator will be used in the classroom, and it seems unlikely that conceptual uses such as those outlined above will routinely take place without teacher guidance. This in turn requires the teacher to have developed a perspective that the calculator is a tool for learning, not merely a tool for computation, as well as be provided with good examples of how to use the calculator in such a way. For this to occur, good professional development for teachers is critical: teachers need mastery of scientific calculator skills to their level as well as good resources to enable them to provide opportunities for students to develop mathematical concepts using scientific calculators. Without such support for teachers, both evidence and experience suggest that the technology will not be used effectively.

The scientific calculator may be used individually or in pairs or with an emulator for class discussion. There needs to be no mismatch between the students' understanding of the purpose of the lesson and the teachers'. In the early stages of concept building, when calculators may be unfamiliar to students, some technical features have to learned; these can be learned with a combination of teacher guidance and students' working together. Both teachers and students need to become really familiar with technology so that the focus can be on learning mathematics rather than on keystrokes

The use of the calculator gives the opportunity for students to engage in real life problems, problem solving and authentic tasks in groups or individually. Many teachers have suggested that conceptual development is helped by students discussing their work with others, either in small groups or in the whole classroom, which of course requires teachers to create a classroom environment for such conversations to take place.

## 5. Conclusion

While the original scientific calculators of forty years ago were mostly designed to render computations efficient for scientists and engineers, much has happened since then to refine the handheld calculator into an educational tool for learning mathematics. In many affluent countries, awash with sophisticated technologies and adequate resources, the scientific calculator has faded from educational view, yet it remains a potentially powerful tool for students to develop concepts, provided teachers are given sufficient help to support such work. This is particularly so in the developing world, in which both financial and human resources are less readily available. In this paper, we have illustrated this argument by outlining two of the ways in which modern scientific calculators can be used for sound educational purposes, focusing on their capacity for multiple representation and their ability to represent key mathematical ideas numerically.

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