Rich, Sturmian & trapezoidal words

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Formally:

- A *word* is a finite or infinite sequence of symbols (*letters*) taken from a non-empty countable set $\mathcal{A}$ (*alphabet*).

Examples

- 001
- $(001)^\infty = 001001001001001001001001001\cdots$
- $11001111000110111011110011011100101111111101\cdots$
- $10010211012222010211002111110221222220111 2012\cdots$
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  Basic measure: number of distinct blocks (factors) of each length occurring in the word.

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Conjecture: $C_{\mathbf{x}}(n) = 2^n$ for all n as it is believed $\sqrt{2}$ is *normal* in base 2.

Complexity & Periodicity

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- Numerous equivalent definitions & characterisations . . .

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An infinite word $\mathbf{w}$ is Sturmian if and only if

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- A Sturmian word over the alphabet $\{a, b\}$ contains either aa or bb, but not both.

Constructing Sturmian words

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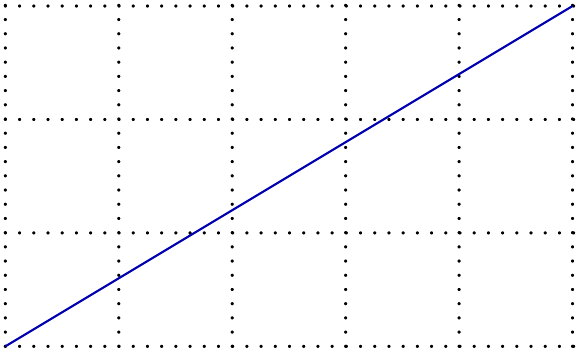
- Words over a 2-letter alphabet $\{a, b\}$ that are factors of (infinite) Sturmian words are called *finite Sturmian words* – they are the cyclic shifts of *Christoffel words* obtained via the following construction.
- Using a similar construction we obtain infinite Sturmian words.

Christoffel words: Construction by example

Lower Christoffel word of slope $\frac{3}{5}$



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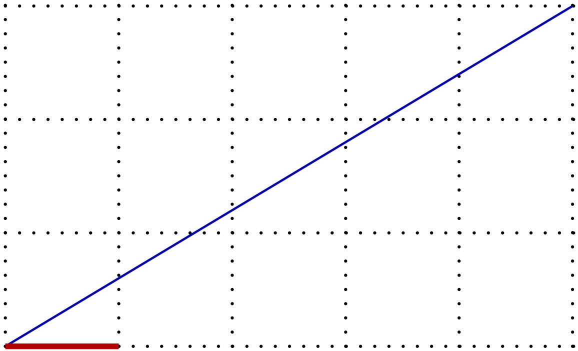

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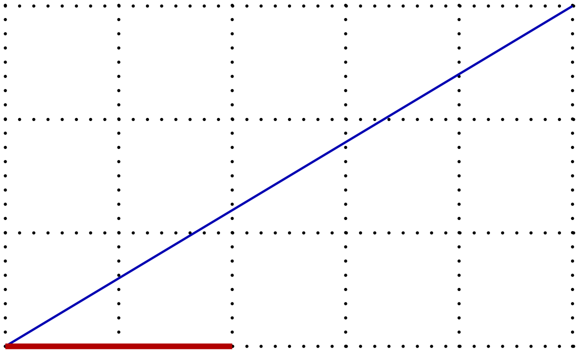

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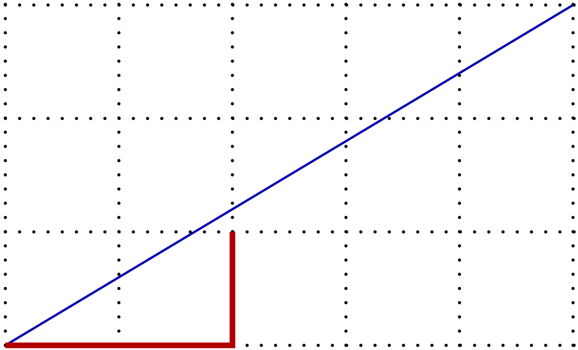

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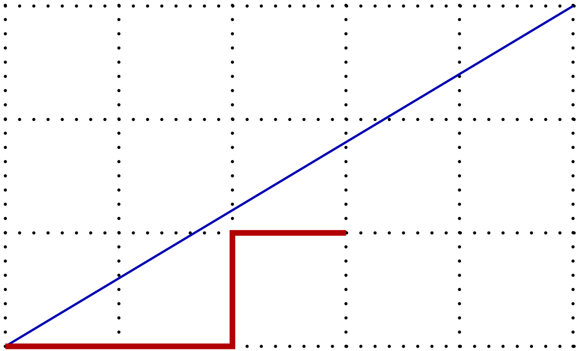

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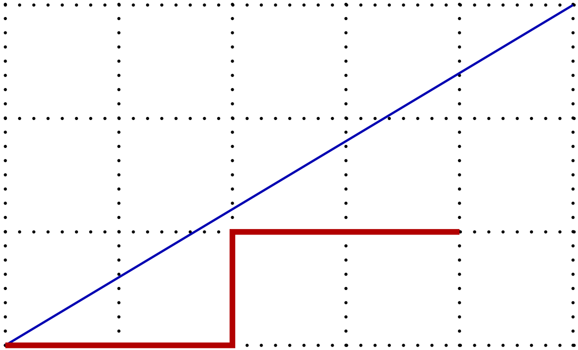

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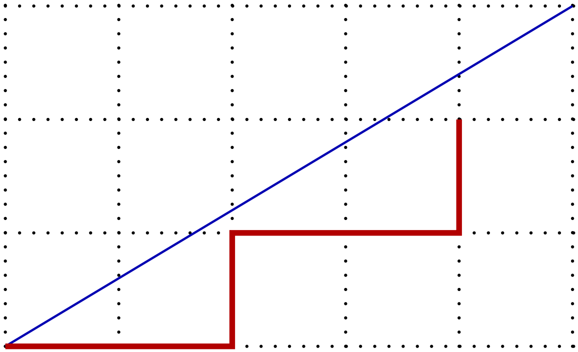

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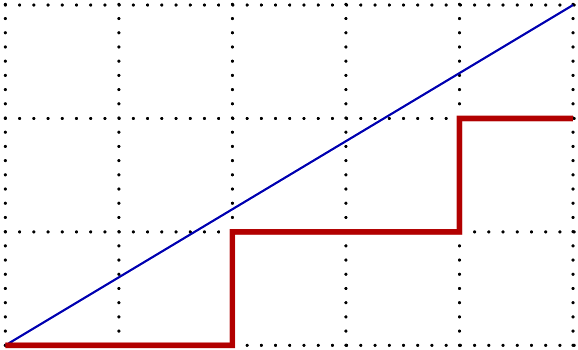

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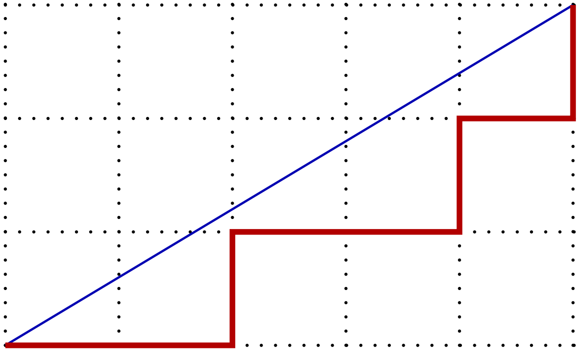

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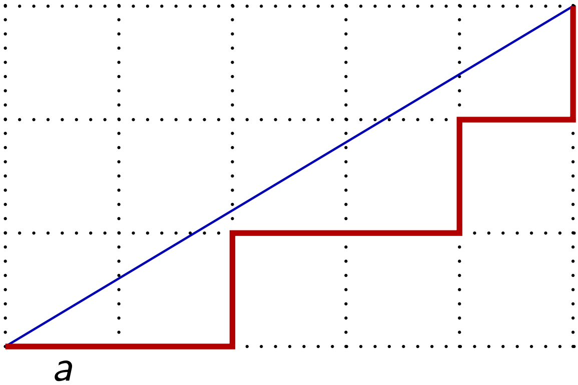

$$L(3,5) = a$$

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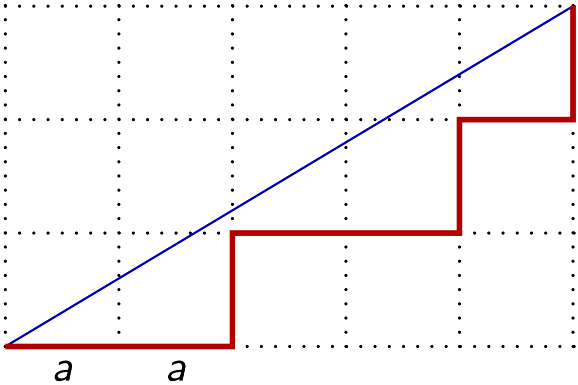

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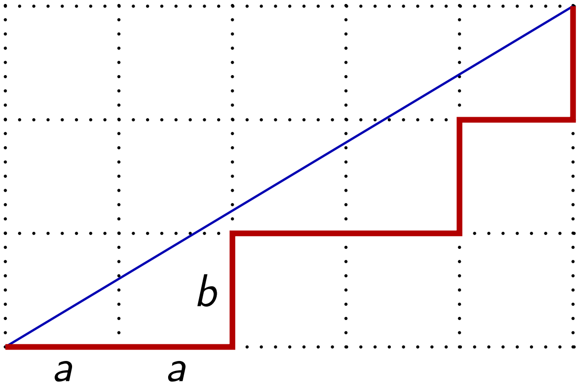

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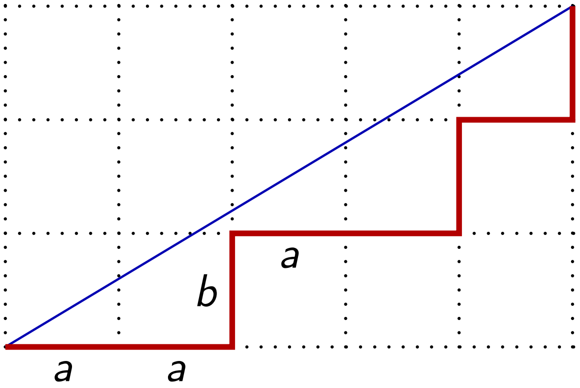

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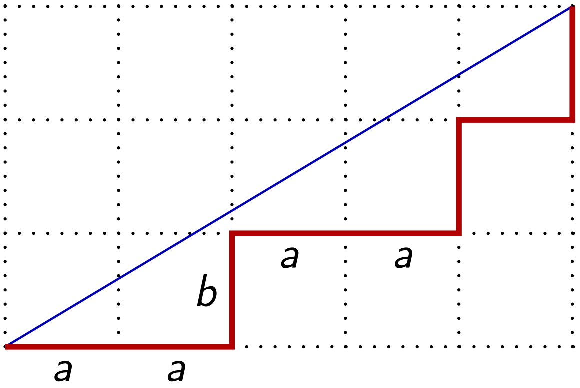

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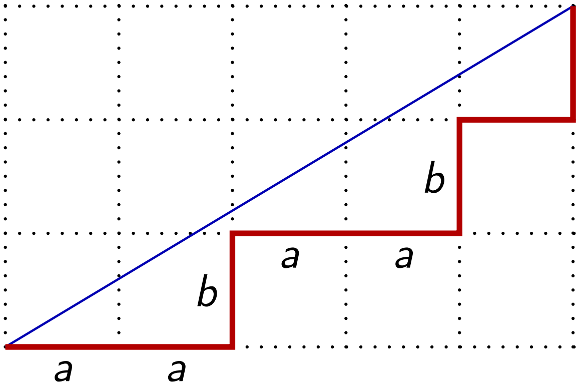

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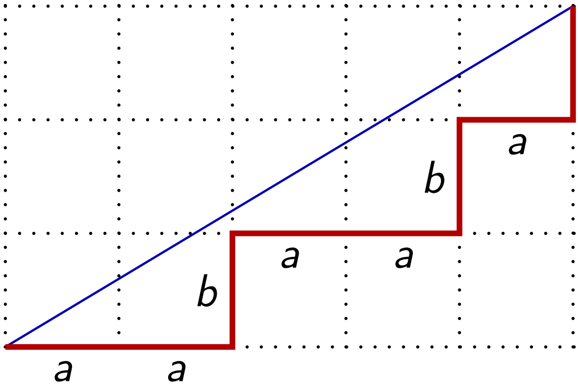

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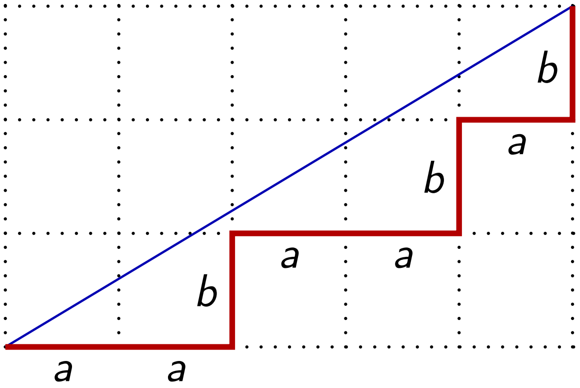

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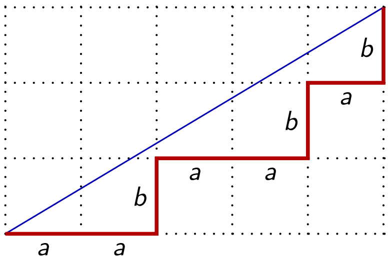

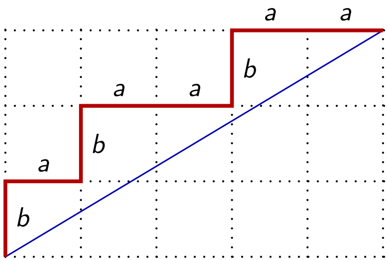

L(3,5) = aabaabab

U(3,5) = babaabaa

From Christoffel words to Sturmian words

Sturmian words: Obtained **similarly** by replacing the line segment by a half-line:

$$y = \alpha x + \rho \text{ with irrational } \alpha \in (0,1),\ \rho \in \mathbb{R}.$$

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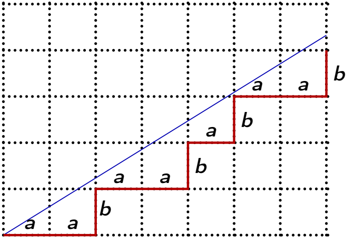
Example: $y = \frac{\sqrt{5}-1}{2}x \longrightarrow$ *Fibonacci word*



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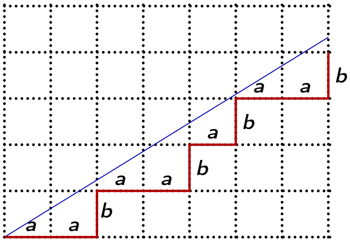

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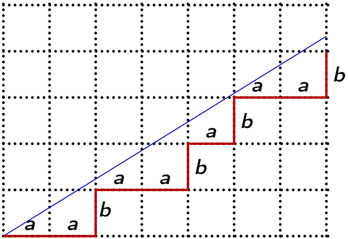

- $\boldsymbol{f} = abaababaabaababaaba\cdots$ (note: disregard 1st a in construction)
- *Standard Sturmian word* of slope $\frac{\sqrt{5}-1}{2}$, golden ratio conjugate

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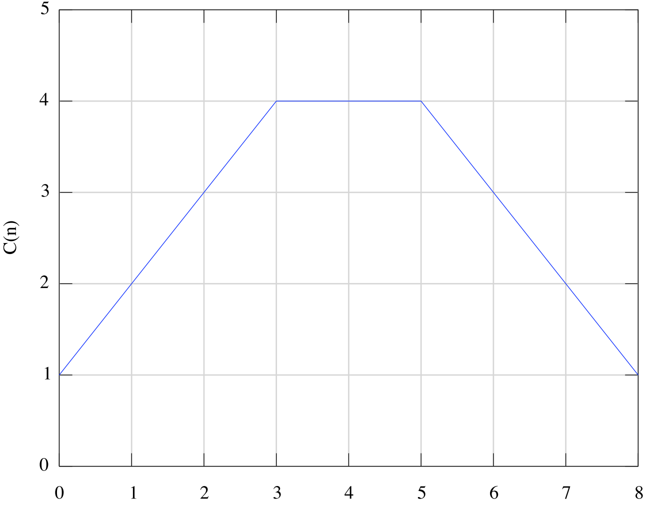
So if we set $D_w(n) = C_w(n+1) - C_w(n)$ for each n with $0 \leq n \leq |w| - 1$, then the word $D_w(0)D_w(1)\cdots D_w(|w|-1)$ takes the form $1^r 0^s (-1)^r$.

Example

Graph of the factor complexity of the Christoffel word $L(3,5) = aabaabab$



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- F. D'Alessandro (2002): classified all non-Sturmian trapezoidal words.

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- abc is rich, but $(abc)^\omega = abcabcabc\cdots$ is **not** rich.

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In short, a word is rich if and only if all complete returns to palindromes are palindromes.

More General Examples

Rich words have appeared in many different contexts; they include:

- Sturmian and episturmian words
  Droubay-Justin-Pirillo (2001)
  Anne-Zamboni-Zorca (2005)
  Bucci-De Luca-G.-Zamboni (2008)
- Complementation-symmetric Rote sequences
  Allouche-Baake-Cassaigne-Damanik (2003) + Bucci-De Luca-G.-Zamboni (2008)
- Symbolic codings of trajectories of symmetric interval exchange transformations – Ferenczi-Zamboni (2008)
- A certain class of words associated with β-expansions where β is a simple Parry number
  Ambrož-Frougny-Masáková-Pelantová (2006) + Bucci-De Luca-G.-Zamboni (2008)
- Infinite words with "abundant palindromic prefixes"
  Introduced by Fischler in 2006 in relation to Diophantine approximation

A Connection Between Palindromic & Factor Complexity

Allouche-Baake-Cassaigne-Damanik, 2003: for any aperiodic infinite word $\boldsymbol{w}$,

$$P_{\boldsymbol{w}}(n) \leq \frac{16}{n}\, C_{\boldsymbol{w}}\left(n + \left\lfloor \frac{n}{4} \right\rfloor\right) \quad \text{for all } n \in \mathbb{N}.$$

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Baláži-Masáková-Pelantová, 2007: for any uniformly recurrent infinite word $\boldsymbol{w}$ with $\mathcal{F}(\boldsymbol{w})$ closed under reversal,

$$P_{\boldsymbol{w}}(n) + P_{\boldsymbol{w}}(n+1) \leq C_{\boldsymbol{w}}(n+1) - C_{\boldsymbol{w}}(n) + 2 \quad \text{for all } n \in \mathbb{N}. \qquad (*)$$

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Bucci-De Luca-G.-Zamboni, 2008: infinite words $\boldsymbol{w}$ for which $P_{\boldsymbol{w}}(n) + P_{\boldsymbol{w}}(n+1)$ reaches the upper bound in $(*)$ for every n are **rich** . . .

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Theorem (Bucci-De Luca-G.-Zamboni 2008)

For any infinite word $\boldsymbol{w}$ with set of factors $\mathcal{F}(\boldsymbol{w})$ closed under reversal, the following conditions are equivalent:

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- Infinite words over $\{a, b\}$ with factors closed under both complementation and reversal, and such that $C(n) = 2n$ for all n.

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- Hence $P(n) + P(n+1) = 4 = C(n+1) - C(n) + 2$ for all $n \Rightarrow$ RICH.

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Finite Case

Using completely different methods . . .

Theorem (de Luca-G.-Zamboni)

For any finite word w, the following two conditions are equivalent:

i) w is a rich palindrome;

ii) $P_w(n) + P_w(n+1) = C_w(n+1) - C_w(n) + 2$ for all n, $0 \leq n \leq |w|$.

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  Example: $(aacbccbcacbc)^{\omega} = aacbccbcacbc\,aacbccbcacbc \cdots$
- action of morphisms on (almost) rich words

More Stuff on Rich Words

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Open Problems

- Characterisation of morphisms that preserve (almost) richness
- Enumeration of rich words

Dammit, I'm mad!

U R 2 R U?

* Both phrases are (rich) palindromes! *