Powers in a class of $\mathcal{A}$-strict standard episturmian words

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  - Share many properties with Sturmian words.
  - Include the well-known *Arnoux-Rauzy sequences*.
  - Introduced by Droubay, Justin, and Pirillo (2001).

Terminology and notation

Let $\mathcal{A}$ denote a *finite alphabet* and let $u = x_1 x_2 \cdots x_m$ where each $x_i \in \mathcal{A}$.

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- For $0 \leq j \leq m-1$, the *j-th conjugate* of u is the word

$$C_j(u) := x_{j+1} x_{j+2} \cdots x_m x_1 x_2 \cdots x_j$$

and we define

$$\mathcal{C}(u) := \{C_j(u) \ : \ 0 \leq j \leq |u| - 1\},$$

the *conjugacy class* of u.

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- A finite word w is said to have a non-trivial *integer power* in $\mathbf{x}$ if

$$w^p = \underbrace{ww \cdots w}_{p}$$

is a factor of $\mathbf{x}$ for some integer $p \geq 2$.

Aim

- Explicitly determine all integer powers occurring in episturmian words.
- This has been done for Sturmian words by Damanik & Lenz (2003).
- We do this for a restricted class of episturmian words.

Sturmian words

Definition

An infinite word $\mathbf{s}$ over $\{a, b\}$ is *Sturmian* if there exist real numbers α, $\rho \in [0, 1]$ such that $\mathbf{s}$ is equal to one of the following two infinite words:

$$s_{\alpha,\rho},\ s'_{\alpha,\rho} : \mathbb{N} \to \{a, b\}$$

defined by

$$s_{\alpha,\rho}(n) = \begin{cases} a & \text{if } \lfloor (n+1)\alpha + \rho \rfloor - \lfloor n\alpha + \rho \rfloor = 0, \\ b & \text{otherwise;} \end{cases}$$

$$(n \geq 0)$$

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Sturmian & episturmian words

- A *Sturmian word* is:
  - *aperiodic* if α is irrational;
  - *periodic* if α is rational;
  - *characteristic* (or *standard*) if $\rho = \alpha$.

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- **Episturmian words**
  - An infinite word $\mathbf{t}$ over $\mathcal{A}$ is *episturmian* if:
    - $\Omega(\mathbf{t})$ is *closed under reversal*, and
    - $\mathbf{t}$ has *at most one right special factor of each length*.

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  - An episturmian word is *standard* if all of its *left special factors are prefixes of it*.
  - Sturmian words are exactly the aperiodic episturmian words over a 2-letter alphabet.

Standard episturmian words

- Let $\mathbf{t}$ be a standard episturmian word over $\mathcal{A}$ and let

$$u_1 = \varepsilon, \ u_2, \ u_3, \ u_4, \ \ldots$$

be the infinite sequence of its palindromic prefixes.

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- $\exists$ an infinite word $\Delta(\mathbf{t}) = x_1x_2x_3\cdots$ ($x_i \in \mathcal{A}$) such that

$$\boxed{u_{n+1} = (u_nx_n)^{(+)}, \quad n \in \mathbb{N}^+}$$

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- $\mathbf{t} = \lim_{n\to\infty} u_n$

Strict episturmian words

- **Notation**
  - Ult$(\mathbf{t})$: set of letters occurring infinitely often in $\mathbf{t}$
  - Alph$(\mathbf{t}) := \Omega(\mathbf{t}) \cap \mathcal{A}$, the *alphabet* of $\mathbf{t}$

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- **Definition**
  - A standard episturmian word $\mathbf{t}$ over $\mathcal{A}$, or any equivalent (episturmian) word, is said to be *$\mathcal{B}$-strict* (or *k-strict* if $|\mathcal{B}| = k$) if

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- **Remarks**
  - The k-strict episturmian words have *complexity* $(k-1)n + 1$ for each $n \in \mathbb{N}$.
  - Such words are exactly the k-letter *Arnoux-Rauzy sequences*.
  - Example: *k-bonacci word*, $k \geq 2$.

A class of strict standard episturmian words

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- For any standard episturmian word $\mathbf{t}$,

$$\Delta(\mathbf{t}) = a_1^{d_1} a_2^{d_2} \cdots a_k^{d_k} a_1^{d_{k+1}} a_2^{d_{k+2}} \cdots a_k^{d_{2k}} a_1^{d_{2k+1}} \cdots ,$$

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  where each $d_i \geq 0$.
- We restrict our attention to the case when all $d_i > 0$.
- Let $\mathbf{s}$ be the k-strict standard episturmian word with directive word:

$$\Delta(\mathbf{s}) = a_1^{d_1} a_2^{d_2} \cdots a_k^{d_k} a_1^{d_{k+1}} a_2^{d_{k+2}} \cdots a_k^{d_{2k}} a_1^{d_{2k+1}} \cdots , \quad d_i > 0.$$

Example

- Let $\alpha \in (0,1)$ be irrational with $\alpha = [0; 1+d_1, d_2, d_3, \ldots]$, $d_1 \geq 1$.
- The *characteristic Sturmian word* c_α over $\{a,b\}$ has directive word

$$\Delta(c_\alpha) = a^{d_1} b^{d_2} a^{d_3} b^{d_4} a^{d_5} \cdots .$$

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- $\Delta(\mathbf{s})$ resembles $\Delta(c_\alpha)$.
- One can prove that $\mathbf{s} = \lim_{n\to\infty} s_n$ where the sequence $(s_n)_{n\geq 1-k}$ is defined by

$$s_{1-k} = a_2, \quad s_{2-k} = a_3, \quad \ldots\ , \quad s_{-1} = a_k, \quad s_0 = a_1,$$

$$s_n = s_{n-1}^{d_n} s_{n-2}^{d_{n-1}} \cdots s_0^{d_1} a_{n+1}, \quad 1 \leq n \leq k-1,$$

$$s_n = s_{n-1}^{d_n} s_{n-2}^{d_{n-1}} \cdots s_{n-k+1}^{d_{n-k+2}} s_{n-k}, \quad n \geq k.$$

Powers

- Let $p \geq 2$ be an integer.
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Key tools in our analysis of powers occurring in $\mathbf{s}$:

- canonical decompositions of $\mathbf{s}$ in terms of its building blocks s_n;
- a generalization of singular words.

Singular words

- The set of factors of length $|s_n|$ in c_α is given by

$$\{\text{all conjugates of } s_n\} \cup \{w_n\}$$

where w_n is called the *n-th singular factor* of c_α.

[Wen and Wen (1994), Melançon (1999), Cao and Wen (2003)]

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- Each set Ω_n^i is closed under reversal.
- If $w \in \Omega_n^i$ then w is called a *singular n-word of the i-th kind.*
- Such words play a key role in our study of powers occurring in $\mathbf{s}$.

Powers occurring in s

- Let $n \in \mathbb{N}^+$ be fixed.
- We define k sets of lengths between $|s_n|$ and $|s_{n+1}|$:

$$\mathcal{D}_1(n) := \{r|s_n| \ : \ 1 \le r \le d_{n+1}\},$$

$$\mathcal{D}_i(n) := \{|s_n^r s_{n-1}^{d_n} \cdots s_{n+2-i}^{d_{n+3-i}} s_{n+1-i}| \ : \ 1 \le r \le d_{n+1}\}, \quad 2 \le i \le k-1,$$

$$\mathcal{D}_k(n) := \{|s_n^r s_{n-1}^{d_n} \cdots s_{n+2-k}^{d_{n+3-k}} s_{n+1-k}| \ : \ 1 \le r \le d_{n+1} - 1\}.$$

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Let $\mathcal{D}_n = \bigcup_{i=1}^{k} \mathcal{D}_i(n)$.

- Suppose $w \prec \mathbf{s}$ and let $p \geq 2$ be an integer. Then,

$$\boxed{w^p \prec \mathbf{s} \quad \Rightarrow \quad |w| \in \mathcal{D}_n \text{ for some } n.}$$

Squares

- In s, successive occurrences of a singular word are *positively separated.*

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- In s, successive occurrences of a singular word are *positively separated.*
- Consequently:

  Lemma: *Suppose* $u^2 \prec \mathbf{s}$ *with* $|u| \in \mathcal{D}_n$*. Then*

$$w \not\prec u \quad \text{if} \quad w \in \Omega^1_{n+1-i} \text{ for some } i \in [1, k-1].$$

Squares

- In $\mathbf{s}$, successive occurrences of a singular word are *positively separated.*
- Consequently:

  Lemma: *Suppose* $u^2 \prec \mathbf{s}$ *with* $|u| \in \mathcal{D}_n$. *Then*

  $$w \not\prec u \quad \textit{if} \quad w \in \Omega^1_{n+1-i} \textit{ for some } i \in [1, k-1].$$

  - That is, u does not contain a singular $(n+1-i)$-word of the first kind for any $i \in [1, k-1]$.

Squares, cubes, and higher powers

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  - Then:

$$w^p \prec \mathbf{s} \quad \Longleftrightarrow \quad w \text{ is one of the first } |s_n| \text{ conjugates of } (s_n)^r.$$

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  (The lengths $|s_n|$ are the *k-bonacci numbers*.)
- If $w^p \prec \boldsymbol{\eta}_k$, then

  $|w| = |s_n| + |s_{n-1}| + \cdots + |s_{n+1-i}|$ for some $n \in \mathbb{N}$ and $i \in [1, k-1]$.

- Our main results reveal that, in $\boldsymbol{\eta}_k$,
  - $(a_1)^2$ is the unique square of length 2;
  - all conjugates of s_n have a square;
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  - only some conjugates of s_n have a cube.
- There are no other integer powers in $\boldsymbol{\eta}_k$.
  - In particular, the k-bonacci word is 4-power free.

Open problems

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    - Generalize other results on powers in Sturmian words to episturmian words by expanding on the above work.
    - Search for further uses of generalized singular words . . .

Open problems (cont.)

- **Decompositions**
  - Establish *singular decompositions* of episturmian words, i.e., factorizations with respect to occurrences of singular words (or *Lyndon words* perhaps).

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  - Establish *singular decompositions* of episturmian words, i.e., factorizations with respect to occurrences of singular words (or *Lyndon words* perhaps).
  - In particular, can one describe precisely where palindromes occur in a given episturmian word?

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Consider the *infinite Fibonacci word*

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- **Note:** n-th block ($n \geq -1$) is called the *n-th singular word* of $\mathbf{f}$.
  - It is a palindrome.
  - Its length is F_n, the *n-th Fibonacci number* given by:

$$F_{-1} = F_0 = 1, \quad F_n = F_{n-1} + F_{n-2}, \quad n \geq 1.$$

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- See "*Occurrences of palindromes in characteristic Sturmian words*", by A. Glen (2006).

Another problem . . .

Terminology

- A finite word $w \in \mathcal{A}^+$ is called an *overlap* if

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- Example: `alfalfa`
- A finite or infinite word is *overlap-free* if it does not contain an overlap as a factor.

Thue-Morse word $\mathbf{t}$

Definition

- Infinite overlap-free word over a 2-letter alphabet $\{0,1\}$ given by:

$$\mathbf{t} = \lim_{n\to\infty} \mu^n(0) = 0110100110010110\cdots$$

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- Also

$$\mathbf{t} = (s_2(n) \bmod 2)_{n \geq 0}$$

  where $s_m(n)$ denotes the sum of the digits in the base-m representation of n.

Generalized Thue-Morse sequences

Allouche and Shallit (2000):

- Let $k \geq 1$, $m \geq 2$ be integers.
- Then the infinite word

$$\mathbf{t}_{m,k} := (s_m(n) \bmod k)_{n \geq 0}$$

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- **Open problem:**
  - Determine the largest (fractional) power that occurs in $\mathbf{t}_{m,k}$.

Problem (cont.)

- **Fractional powers:**
  - Suppose $w = (uv)^n u$ where $u, v \in \mathcal{A}^+$. Then

$$w = z^r, \quad \text{where } z = uv \text{ and } r := n + |u|/|z|.$$

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  - $\mathbf{x}$ has *bounded index* if $\exists\, d \in \mathbb{R}$ such that $\sup_{w \in \Omega(\mathbf{x})} \text{ind}(w) \leq d$.
    - d is called the *critical exponent* of $\mathbf{x}$.

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  - $\mathbf{t}_{m,k}$ contains the square of a letter $\Leftrightarrow$ gcd$(m-1,k)=1$.
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- Recall: We wish to determine the critical exponent of $\mathbf{t}_{m,k}$.
  - Only need to consider the case $m > k$ since 2 is sharp for $m \leq k$.
  - First show that every non-empty factor of $\mathbf{t}_{m,k}$ has finite index.