

## FOURIER MULTIPLIERS FOR LOCAL HARDY SPACES ON CHÉBLI-TRIMÈCHE HYPERGROUPS

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**ABSTRACT.** In this paper we consider Fourier multipliers on local Hardy spaces  $\mathbf{h}^p$  ( $0 < p \leq 1$ ) for Chébli-Trimèche hypergroups. The molecular characterization is investigated which allows us to prove a version of Hörmander's multiplier theorem.

The theory of Fourier multipliers is well developed on euclidean spaces, with various results having been established to give sufficient conditions for a multiplier operator to be bounded on the Lebesgue spaces  $L^p$  ( $p > 1$ ) or Hardy spaces  $H^p$  ( $0 < p \leq 1$ ). Among these are Hörmander's multiplier theorem and its variants. Over the past twenty years considerable effort has been made to extend the classical Fourier multiplier theory to groups and hypergroups (see [W], [FS], [AI], [FX] on Lie groups, [CS], [ST], [K], [An] on noncompact symmetric spaces and [S] on Bessel-Kingman hypergroups). In the consideration of this problem a dichotomy is emerging, based on the growth of the volume of balls centered at the identity as their radii become large (polynomial or exponential growth). While in the case of polynomial growth the condition on a multiplier is similar to that on euclidean spaces (see [AI], [FX], [FS] and [W]), some holomorphy of the multiplier is necessary for the operator to be bounded on  $L^p$  ( $p > 1$ ) when the volumes of balls grow exponentially. In the latter situation, the  $L^p$  Fourier multiplier has to be an analytic function having a holomorphic extension to a prescribed tube, the size of which depends on  $p$  (see [CS], [An] and [BX3]). This is a consequence of the "holomorphic extension" property of Fourier transforms of  $L^p$ -functions ( $1 \leq p < 2$ ). It turns out that the holomorphic extension property of the Fourier multiplier corresponds roughly to exponential decay of the kernel.

Most of the work up till now on Fourier multipliers on Lie groups and symmetric spaces has only been concerned with  $L^p$ -boundedness for  $p > 1$ , and the  $H^p$ -multiplier results in [FS] were only proved on stratified Lie groups which are of polynomial growth. A version of Hörmander's multiplier theorem was established in [S] for  $L^p$ -functions ( $p > 1$ ) on Bessel-Kingman hypergroups, a particular class of Chébli-Trimèche hypergroups with polynomial growth. For general Chébli-Trimèche hypergroups the  $L^p$ -Fourier multipliers were investigated in [BX3].

A natural question that arises is whether we can extend the  $L^p$  ( $p > 1$ ) Fourier multiplier theorem of Hörmander to the case  $0 < p \leq 1$  for those hypergroups, Lie groups and symmetric spaces that are of exponential growth. There is indeed a natural candidate for

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such a generalization, using the local Hardy spaces  $\mathbf{h}^p$  (see Section 3 for the definition). In [K] there are some results on Fourier multipliers for the local Hardy space  $\mathbf{h}^1$  on noncompact symmetric spaces of rank 1 (which is of exponential growth), but the conditions on the multipliers are not as sharp and natural as in the classical Hörmander multiplier theorem. Moreover, the approach in [K] does not work for general  $\mathbf{h}^p$  when  $p < 1$ . In fact the molecules defined in [K] are not appropriate to handle  $\mathbf{h}^p - \mathbf{h}^p$  boundedness of a Fourier multiplier operator when  $p < 1$ . This is because the generalized translation of a polynomial of degree  $\geq 1$  need not be a polynomial. We have modified the definition of molecule to cater for this new phenomenon (see Definition 2.4 and Remark 2.5). To the best of our knowledge, nobody has examined systematically the local Hardy spaces  $\mathbf{h}^p$  and Fourier multipliers for  $\mathbf{h}^p$  on Chébli-Trimèche hypergroups and noncompact symmetric spaces (other than in [BX2]).

In this paper we establish a version of Hörmander's multiplier theorem for the local Hardy spaces  $\mathbf{h}^p$  ( $0 < p \leq 1$ ) on Chébli-Trimèche hypergroups with exponential growth. Because of the exponential volume growth and the generalized convolution on the hypergroup, the standard constructions do not apply. Many basic facts relying on the structure of a euclidean space are largely unavailable; the Fourier transform on hypergroups is far less well understood than on euclidean spaces, and furthermore there is no "convenient" dilation structure on hypergroups. Our method is a combination of the techniques for euclidean spaces and for noncompact symmetric spaces, and indeed the approach used here can be easily applied to noncompact symmetric spaces.

The paper is organized as follows. The basic Fourier analysis on hypergroups and some useful estimates for characters are given in Section 1. In Section 2 we give an appropriate definition of (local) molecules and investigate the molecular characterization of local Hardy spaces  $\mathbf{h}^p$  for  $0 < p \leq 1$ . Finally in Section 3 we use this molecular characterization to obtain a version of Hörmander's multiplier theorem.

**1. Preliminaries on Chébli-Trimèche hypergroups.** We begin by recalling some basic facts of Fourier analysis on Chébli-Trimèche hypergroups; for a general reference see [BH].

Throughout the paper we denote by  $(\mathbf{R}_+, *(A))$  the Chébli-Trimèche hypergroup associated with a function  $A$  that is continuous on  $\mathbf{R}_+$ , twice continuously differentiable on  $\mathbf{R}_+^* = ]0, \infty[$ , and satisfies the following conditions:

- (1.1)  $A(0) = 0$  and  $A(x) > 0$  for  $x > 0$ ;
- (1.2)  $A$  is increasing and unbounded;
- (1.3)  $\frac{A'(x)}{A(x)} = \frac{2\alpha+1}{x} + B(x)$  on a neighbourhood of 0 where  $\alpha > -\frac{1}{2}$  and  $B$  is an odd  $C^\infty$ -function on  $\mathbf{R}$ ;
- (1.4)  $\frac{A'(x)}{A(x)}$  is a decreasing  $C^\infty$ -function on  $\mathbf{R}_+^*$ , and hence  $\rho := \frac{1}{2} \lim_{x \rightarrow +\infty} \frac{A'(x)}{A(x)} \geq 0$  exists.

In addition we assume that for each  $k \in \mathbf{N}_0$ ,  $(\frac{A'(x)}{A(x)})^{(k)}$  is bounded for large  $x \in \mathbf{R}_+$ .

The hypergroup  $(\mathbf{R}_+, *(A))$  is noncompact and commutative with neutral element 0 and the identity mapping as the involution. Haar measure on  $(\mathbf{R}_+, *(A))$  is given by

$m := A\lambda_{\mathbf{R}_+}$  where  $\lambda_{\mathbf{R}_+}$  is Lebesgue measure on  $\mathbf{R}_+$ . The growth of the hypergroup is determined by the number  $\rho$  in (1.4). If  $\rho > 0$  then (1.4) implies that  $A(x) \geq A(1)e^{2\rho(x-1)}$  for  $x \geq 1$  and so the hypergroup is of exponential growth. Otherwise we say that the hypergroup is of subexponential growth. In this paper we restrict ourselves to Chébli-Trimèche hypergroups of exponential growth.

Let  $L = L_A$  be the differential operator defined for  $x > 0$  by

$$(1.5) \quad Lf(x) = -f''(x) - \frac{A'(x)}{A(x)}f'(x)$$

for each function  $f$  twice differentiable on  $\mathbf{R}_+^*$ . The multiplicative functions on  $(\mathbf{R}_+, *(A))$  coincide with all the solutions  $\varphi_\lambda (\lambda \in \mathbf{C})$  of the differential equation

$$(1.6) \quad L\varphi_\lambda(x) = (\lambda^2 + \rho^2)\varphi_\lambda(x), \quad \varphi_\lambda(0) = 1, \quad \varphi'_\lambda(0) = 0,$$

and the dual space  $\mathbf{R}_+^\wedge$  can be identified with the parameter set  $\mathbf{R}_+ \cup i[0, \rho]$ .

For  $0 < p \leq \infty$  the Lebesgue space  $L^p(\mathbf{R}_+, Adx)$  is defined as usual, and we denote by  $\|f\|_{p,A}$  the  $L^p$ -norm of  $f \in L^p(\mathbf{R}_+, Adx)$ . For  $f \in L^1(\mathbf{R}_+, Adx)$  the Fourier transform of  $f$  is given by

$$(1.7) \quad \hat{f}(\lambda) = \int_{\mathbf{R}_+} f(x)\varphi_\lambda(x)A(x) dx.$$

**THEOREM 1.8 (LEVITAN-PLANCHEREL; see [BH, THEOREM 2.2.13]).** *There exists a unique nonnegative measure  $\sigma$  on  $\mathbf{R}_+^\wedge$  with support  $[\rho^2, \infty[$  such that the Fourier transform induces an isometric isomorphism from  $L^2(\mathbf{R}_+, Adx)$  onto  $L^2(\mathbf{R}_+^\wedge, \sigma)$ , and for any  $f \in L^1(\mathbf{R}_+, Adx) \cap L^2(\mathbf{R}_+, Adx)$*

$$\int_{\mathbf{R}_+} |f(x)|^2 A(x) dx = \int_{\mathbf{R}_+^\wedge} |\hat{f}(\lambda)|^2 \sigma(d\lambda).$$

The inverse is given by

$$f(x) = \int_{\mathbf{R}_+^\wedge} \hat{f}(\lambda)\varphi_\lambda(x)\sigma(d\lambda).$$

To determine the Plancherel measure  $\sigma$  we must place a further (growth) restriction on  $A$ . A function  $f$  is said to satisfy condition (H) if for some  $a > 0$ ,  $f$  can be expressed as

$$f(x) = \frac{a^2 - \frac{1}{4}}{x^2} + \zeta(x)$$

for all large  $x$  where

$$\int_{x_0}^{\infty} x^{\gamma(a)} |\zeta(x)| dx < \infty$$

for some  $x_0 > 0$  and  $\zeta(x)$  is bounded for  $x > x_0$ ; here  $\gamma(a) = a + \frac{1}{2}$  if  $a \geq \frac{1}{2}$  and  $\gamma(a) = 1$  otherwise. For  $x > 0$  we put

$$G(x) := \frac{1}{4} \left( \frac{A'(x)}{A(x)} \right)^2 + \frac{1}{2} \left( \frac{A'(x)}{A(x)} \right)' - \rho^2.$$

THEOREM 1.9 (see [BX1, PROPOSITION 3.17] or [T2]). *Suppose that  $G$  satisfies condition (H) together with one of the following conditions:*

- (i)  $a > \frac{1}{2}$ ;
- (ii)  $a \neq |\alpha|$ ;
- (iii)  $a = \alpha \leq \frac{1}{2}$  and  $\int_0^\infty t^{\frac{1}{2}-\alpha} \zeta(t) \varphi_0(t) A(t)^{\frac{1}{2}} dt \neq -2\alpha \sqrt{M_A}$  or  $\int_0^\infty t^{\alpha+\frac{1}{2}} \zeta(t) \varphi_0(t) A(t)^{\frac{1}{2}} dt = 0$  where  $M_A := \lim_{x \rightarrow 0^+} x^{-2\alpha-1} A(x)$  and  $\zeta(x) = G(x) + \frac{\frac{1}{2}-a^2}{x^2}$ .

*Then the Plancherel measure  $\sigma$  is absolutely continuous with respect to Lebesgue measure and has density  $|c(\lambda)|^{-2}$  where  $c(\lambda)$  satisfies the following: There exist positive constants  $C_1, C_2, K$  such that for any  $\lambda \in \mathbf{C}$  with  $\text{Im}(\lambda) \leq 0$*

$$C_1 |\lambda|^{a+\frac{1}{2}} \leq |c(\lambda)|^{-1} \leq C_2 |\lambda|^{a+\frac{1}{2}}, \quad |\lambda| \leq K, \quad a > 0$$

$$C_1 |\lambda|^{\alpha+\frac{1}{2}} \leq |c(\lambda)|^{-1} \leq C_2 |\lambda|^{\alpha+\frac{1}{2}}, \quad |\lambda| > K.$$

In the sequel we assume that  $A$  satisfies the conditions in Theorem 1.9. This together with (1.1)–(1.4) with  $\rho > 0$  implies the following result (see [BX1, Lemmas 2.5 and 3.8]):

LEMMA 1.10. *We have*

$$A(x) \sim x^{2\alpha+1} \quad (x \rightarrow 0^+)$$

$$A(x) \sim e^{2\rho x} \quad (x \rightarrow +\infty).$$

Let  $\epsilon_x$  be the unit point mass at  $x \in \mathbf{R}_+$ . For any  $x, y \in \mathbf{R}_+$  the probability measure  $\epsilon_x * \epsilon_y$  is  $m$ -absolutely continuous with

$$(1.11) \quad \text{supp}(\epsilon_x * \epsilon_y) \subset [|x - y|, x + y].$$

The generalized translation  $T_x f$  of a function  $f$  by  $x \in \mathbf{R}_+$  is defined by

$$(1.12) \quad T_x f(y) := \int_{\mathbf{R}_+} f(z) \epsilon_x * \epsilon_y(dz).$$

The convolution of two functions  $f$  and  $g$  is defined by

$$(1.13) \quad f * g(x) = \int_{\mathbf{R}_+} T_x f(y) g(y) A(y) dy.$$

Let us introduce Schwartz functions and distributions on the hypergroup (see [BX4]). For  $0 < p \leq 2$  the generalized Schwartz space  $\mathcal{S}_p(\mathbf{R}_+, *(A))$  consists of the restrictions to  $\mathbf{R}_+$  of all functions in  $\mathcal{S}_p(\mathbf{R})$  where

$$\mathcal{S}_p(\mathbf{R}) := \{g \in C^\infty(\mathbf{R}) : g \text{ is even and } \mu_{k,l}^p(g) < \infty, k, l \in \mathbf{N}_0\}$$

and

$$\mu_{k,l}^p(g) := \sup_{x \in \mathbf{R}_+} (1+x)^l \varphi_0(x)^{-\frac{2}{p}} |g^{(k)}(x)|.$$

For  $0 < p \leq 2$  set

$$F_\delta := \{z \in \mathbf{C} : |\operatorname{Im}(z)| \leq \delta\rho\}$$

where  $\delta = \frac{2}{p} - 1$  and  $\rho > 0$  as in (1.4). Let  $\mathcal{S}(F_\delta)$  denote the extended Schwartz space defined by all functions  $h$  that are even and holomorphic in the interior of  $F_\delta$ , and such that  $h$  together with all its derivatives extend continuously to  $F_\delta$  and satisfy  $\sup_{\lambda \in F_\delta} |h^{(k)}(\lambda)| < \infty$  for any  $k, l \in \mathbf{N}_0$ . Also we denote by  $\mathcal{S}_{\epsilon\rho}(\mathbf{R}_+)$  the space of the restrictions to  $\mathbf{R}_+$  of the functions in  $\mathcal{S}_{\epsilon\rho}(\mathbf{R})$  where

$$\mathcal{S}_{\epsilon\rho}(\mathbf{R}) = \{g \in C^\infty(\mathbf{R}) : g \text{ is even and } \nu_{k,l}^{(\epsilon)}(g) < \infty\}$$

with

$$\nu_{k,l}^{(\epsilon)}(g) := \sup_{t \in \mathbf{R}_+} (1+t)^l e^{\epsilon\rho t} |g^{(k)}(t)|.$$

Notice that  $\mathcal{S}_0 = \mathcal{S}(F_0)$  is the usual Schwartz space on  $\mathbf{R}_+$  and will be denoted by  $\mathcal{S}(\mathbf{R}_+)$ . In the sequel we use  $F$  to denote the Fourier transform on the hypergroup,  $F_0$  the classical Fourier transform and  $A$  the Abel transform (see [T] for the definition).

**THEOREM 1.14** (see [BX4]). *Let  $\delta = \frac{2}{p} - 1$  with  $0 < p \leq 2$ . Then the Fourier transform  $F$  on  $(\mathbf{R}_+, *(A))$  is an isomorphism from  $\mathcal{S}_p(\mathbf{R}_+, *(A))$  to  $\mathcal{S}(F_\delta)$ , the classical Fourier  $F_0$  an isomorphism from  $\mathcal{S}_{\epsilon\rho}(\mathbf{R}_+)$  to  $\mathcal{S}(F_\delta)$ , and the Abel transform  $A$  an isomorphism from  $\mathcal{S}_p(\mathbf{R}_+, *(A))$  to  $\mathcal{S}_{\epsilon\rho}(\mathbf{R}_+)$  satisfying  $Ff = F_0(Af)$ .*

A  $p$ -distribution on  $\mathbf{R}_+$  is a continuous linear functional on  $\mathcal{S}_p(\mathbf{R}_+, *(A))$ ; the totality of  $p$ -distributions on  $\mathbf{R}_+$  is denoted by  $\mathcal{S}'_p(\mathbf{R}_+, *(A))$ . For  $f \in \mathcal{S}'_p(\mathbf{R}_+, *(A))$  we define the Fourier transform of  $f$  by

$$\hat{f}(\phi) := f(F^{-1}\phi), \quad \phi \in \mathcal{S}(F_\delta).$$

By Theorem 1.14,  $\hat{f}$  is well defined as a distribution in  $\mathcal{S}'(F_\delta)$  and  $F$  is continuous on  $\mathcal{S}_p(\mathbf{R}_+, *(A))$ . For  $f \in \mathcal{S}'_p(\mathbf{R}_+, *(A))$  and  $\phi \in \mathcal{S}_p(\mathbf{R}_+, *(A))$  the convolution of  $f$  and  $\phi$  is a  $p$ -distribution defined by

$$f * \phi(\psi) := f(\phi * \psi), \quad \psi \in \mathcal{S}_p(\mathbf{R}_+, *(A)).$$

We now give some useful estimates for characters and their derivatives.

**LEMMA 1.15** (see [C], [AT]). (i) *For each  $\lambda \in \mathbf{C}$ ,  $\varphi_\lambda$  is an even  $C^\infty$ -function and  $\lambda \mapsto \varphi_\lambda(x)$  is holomorphic.*

(ii) *For each  $\lambda \in \mathbf{C}$ ,  $\varphi_\lambda$  has an integral representation (i.e. the Laplace representation)*

$$\varphi_\lambda(x) = \int_{-x}^x e^{(i\lambda - \rho)t} \nu_x(dt), \quad x \in \mathbf{R}_+$$

where  $\nu_x$  is a probability measure on  $\mathbf{R}$  supported in  $[-x, x]$ .

LEMMA 1.16. *Let  $\lambda = \xi + i\eta \in \mathbf{C}$ . Then*

- (i)  $|\varphi_\lambda(x)| \leq e^{|\eta|x} \varphi_0(x)$ ,
- (ii)  $e^{-\rho x} \leq \varphi_0(x) \leq C(1+x)e^{-\rho x}$ .

PROOF. The lemma follows from the Laplace representation of  $\varphi_\lambda$  in Lemma 1.15 and the following estimate given in [AT]:

$$|\varphi_\lambda(x)| \leq C_A(1+x)e^{-\rho x}, \quad x, \lambda \in \mathbf{R}_+. \quad \blacksquare$$

LEMMA 1.17. *Let  $\lambda = \xi + i\eta \in \mathbf{C}$  and  $k \in \mathbf{N}_0$ . Then*

$$|\varphi_\lambda^{(k)}(x)| \leq \begin{cases} C_A(1+|\lambda|)^k e^{|\eta|x}, & |\lambda|x \leq 1, x \leq 1, \\ C_A x A(x)^{-\frac{1}{2}} e^{|\eta|x}, & |\lambda|x \leq 1, x > 1, \\ C_A A(x)^{-\frac{1}{2}} |c(\lambda)| (1+|\lambda|)^k e^{|\eta|x}, & |\lambda|x > 1. \end{cases}$$

We also have the following alternative estimate:

$$|\varphi_\lambda^{(k)}(x)| \leq C_A A(x)^{-\frac{1}{2}} (|\lambda|x)^{\frac{1}{2}-a} |c(\lambda)| (1+|\lambda|)^k e^{|\eta|x}, \quad |\lambda|x \leq 1, x > 1.$$

PROOF. The lemma can be proved similarly to [BX2, Lemma 2.4] using Lemma 1.16. ■

In the sequel we use  $[\beta]$  to denote the largest integer not exceeding  $\beta$ .

LEMMA 1.18. *Let  $\lambda = \xi + i\eta \in \mathbf{C}$  and  $k \in \mathbf{N}_0$ .*

(i) *For all  $x \in \mathbf{R}_+$*

$$|\varphi_\lambda(x)| \leq \begin{cases} C_A x e^{(|\eta|-\rho)x}, & |\eta| < \rho, \\ C_A e^{(|\eta|-\rho)x}, & |\eta| \geq \rho. \end{cases}$$

(ii) *For all  $x > 1$*

$$|\varphi_\lambda^{(k)}(x)| \leq \begin{cases} C_A(1+|\lambda|)^{2[\frac{k+1}{2}]} x e^{(|\eta|-\rho)x}, & |\eta| < \rho, \\ C_A(1+|\lambda|)^{2[\frac{k+1}{2}]} e^{(|\eta|-\rho)x}, & |\eta| \geq \rho. \end{cases}$$

PROOF. Part (i) follows readily from Lemma 1.16 and the Laplace representation of  $\varphi_\lambda$  in Lemma 1.15. Appealing to (1.5) and (1.6) we have

$$\varphi'_\lambda(x) = -\frac{\lambda^2 + \rho^2}{A(x)} \int_0^x \varphi_\lambda(t) A(t) dt$$

and for  $k = 2, 3, \dots$

$$\varphi_\lambda^{(k)}(x) = -\sum_{j=0}^{k-2} \binom{k-2}{j} \left(\frac{A'(x)}{A(x)}\right)^{(j)} \varphi_\lambda^{(k-1-j)}(x) - (\lambda^2 + \rho^2) \varphi_\lambda^{(k-2)}(x).$$

Therefore (ii) follows by induction using (i) and Lemma 1.10 together with our assumption on the derivatives of  $\frac{A'(x)}{A(x)}$ . ■

For an  $m$ -measurable subset  $E$  we denote by  $|E|$  its Haar measure and  $\chi_E$  its characteristic function. For  $x_0 \in \mathbf{R}_+$  and  $r > 0$ ,  $B(x_0, r)$  denotes the open interval  $] \max\{0, x_0 - r\}, x_0 + r[$ . Also in the sequel  $\mathbf{N}_0$  will denote the set of all nonnegative integers. Finally we shall use  $C$  to denote a positive constant whose value may vary from line to line. Dependence of such constants upon parameters of interest will be indicated through the use of subscripts.

**2. The molecular characterization of local Hardy spaces.** In this section we introduce an appropriate definition of (local) molecules and explore the molecular construction of the local Hardy spaces  $\mathbf{h}^p$  (see [TW] for the general theory of molecules on euclidean spaces).

We begin with the definition of the local Hardy space  $\mathbf{h}^p$  and detail its characterization by atomic decomposition (see [BX2]). For  $f \in \mathcal{S}'_1(\mathbf{R}_+, *(A))$  the local heat maximal function is defined by

$$H_0^+ f(x) := \sup_{0 < t \leq 1} |f * h_t(x)|$$

where  $h_t$  is the heat kernel.

DEFINITION 2.1. Let  $0 < p < \infty$ . The local Hardy space  $\mathbf{h}^p = \mathbf{h}^p(\mathbf{R}_+, *(A))$  is defined by

$$\mathbf{h}^p := \{f \in \mathcal{S}'_1(\mathbf{R}_+, *(A)) : H_0^+ f \in L^p(\mathbf{R}_+, A dx)\}.$$

Moreover we introduce the quasi-norm  $\|f\|_{\mathbf{h}^p} := \|H_0^+ f\|_{p,A}$  defining the topology on  $\mathbf{h}^p$ .

We recall that for  $1 < p < \infty$ ,  $\mathbf{h}^p$  coincides with  $L^p(\mathbf{R}_+, A dx)$ . The elementary building blocks of  $\mathbf{h}^p$  are the (local)  $(p, q, s)$ -atoms. Assume throughout that the exponents  $p$  and  $q$  are admissible in the sense that  $0 < p \leq 1$ ,  $1 \leq q \leq \infty$  and  $p < q$ , and put  $s = [(2\alpha + 2)(\frac{1}{p} - 1)]$ .

DEFINITION 2.2. A (local)  $(p, q, s)$ -atom is a function  $a \in L^q(\mathbf{R}_+, A dx)$  such that for some  $x_0 \in \mathbf{R}_+$  and  $r > 0$ ,  $\text{supp}(a) \subset B(x_0, r)$  and

$$\|a\|_{q,A} \leq m(B(x_0, r))^{\frac{1}{q} - \frac{1}{p}}$$

together with the following (local) moment condition: if  $r$  can be chosen not exceeding 1 then

$$\int_0^\infty a(x) x^k A(x) dx = 0$$

for all integers  $k$  satisfying  $0 \leq k \leq s$ .

The following result characterizes  $\mathbf{h}^p$  in terms of atoms.

**THEOREM 2.3** (see [BX2]). *Let  $0 < p \leq 1$ . Then  $f \in \mathbf{h}^p$  if and only if  $f$  can be represented as a linear combination of  $(p, q, s)$ -atoms for any  $1 \leq q \leq \infty, q > p$ :*

$$f = \sum_i \lambda_i a_i$$

where the  $a_i$  are (local)  $(p, q, s)$ -atoms and  $\sum_i |\lambda_i|^p < \infty$ . Moreover there exist two positive constants  $C_1$  and  $C_2$  depending only on  $p$  and  $A$  such that

$$C_1 \left\{ \sum_i |\lambda_i|^p \right\}^{1/p} \leq \|f\|_{\mathbf{h}^p} \leq C_2 \left\{ \sum_i |\lambda_i|^p \right\}^{1/p}.$$

Atoms are very convenient for studying the behaviour of certain operators, like radial maximal operators, on  $\mathbf{h}^p$  (see [BX2]). For example, the continuity of an operator  $T$  can often be proved by estimating  $Ta$  when  $a$  is an atom. However when we consider the  $\mathbf{h}^p - \mathbf{h}^p$  boundedness of an operator  $T$  it is possible that for a general local atom  $a$ ,  $Ta$  may not be an atom itself but has to be decomposed into atoms; indeed in general  $Ta$  will not have compact support. As in the case of euclidean spaces we can find a class of functions more general than atoms which still generate  $\mathbf{h}^p$ . These functions will naturally decompose into atoms, and will be called (local) molecules.

We now introduce the (local) molecules corresponding to the atoms defined above.

**DEFINITION 2.4.** For admissible components  $p, q$  and  $s$  and  $\epsilon > \max\{\frac{s}{2\sigma+2}, \frac{1}{p} - 1\}$  set  $a = 1 - \frac{1}{p} + \epsilon$  and  $b = 1 - \frac{1}{q} + \epsilon$ . A (local)  $(p, q, s, \epsilon)$ -molecule centred at  $x_0 \in \mathbf{R}_+$  is a function  $M \in L^q(\mathbf{R}_+, Adx)$  with  $M(x)|B(x_0, |x - x_0|)^b \in L^q(\mathbf{R}_+, Adx)$  satisfying the conditions

(i)  $\|M\|_{q,A}^{a/b} \|M(x)|B(x_0, |x - x_0|)^b\|_{q,A}^{1-a/b} := N_q(M) < \infty$ , and

(ii) Let  $\sigma$  be the positive number defined by  $|B(x_0, \sigma)|^{\frac{1}{q}-\frac{1}{p}} = \|M\|_{q,A}$ . If  $\sigma < 1$  then for any  $R$  with  $\sigma \leq R \leq 1$ ,

$$\left| \int_{B(x_0,R)} M(x)(x - x_0)^l A(x) dx \right| \leq C_{A,l} \left(\frac{\sigma}{R}\right)^\beta R^l |B(x_0, R)|^{1-\frac{1}{p}}$$

for  $l = 0, 1, \dots, s$  where  $\beta = \min\{a, s + 2 - \frac{1}{p}\}$ .

**REMARK 2.5.** The moment condition enjoyed by a typical molecule on a euclidean space is now replaced by (ii) (which is an immediate consequence of the moment condition in the case of euclidean spaces). In contrast to the case for euclidean spaces, the generalized translation of a polynomial on  $(\mathbf{R}_+, *(A))$  is not necessarily a polynomial. Hence for a local atom  $a$ ,  $Ta$  may not satisfy the moment condition. However (ii) can be satisfied by  $Ta$  for the most important convolution operators  $T$  if  $a$  is a local atom supported in  $B(x_0, r)$  with  $r < 1$ .

For  $x_0 \in \mathbf{R}_+$  and  $\sigma > 0$  we define the following subsets of  $\mathbf{R}_+$ :

$$E_0 := \{x \in \mathbf{R}_+ : |x - x_0| \leq \sigma\} \quad \text{and} \\ E_k := \{x \in \mathbf{R}_+ : 2^{k-1}\sigma < |x - x_0| \leq 2^k\sigma\}, \quad k = 1, 2, \dots$$



and put

$$J_{1,\sigma} := \{k \in \mathbf{N}_0 : 2^k \sigma < 1\} \quad \text{and} \quad J_{2,\sigma} := \{k \in \mathbf{N}_0 : 2^k \sigma \geq 1\}.$$

In the particular case  $0 < x_0 \leq 1$  we associate with  $x_0$  two intervals as follows. First choose the unique integer  $k_0$  such that  $2^{k_0-1} \sigma \leq x_0 < 2^{k_0} \sigma$ , and then define

$$F_{k_0-1} := \begin{cases} [0, x_0 - 2^{k_0-1} \sigma[, & k_0 \geq 2, \\ [0, x_0 + \frac{\sigma}{2}[, & k_0 = 1, \end{cases}$$

$$F_{k_0} := ]x_0 + 2^{k_0-2} \sigma, x_0 + 2^{k_0} \sigma], \quad k_0 \geq 1.$$

For this  $k_0$  we then use  $E_{k_0-1} := F_{k_0-1}$  and  $E_{k_0} := F_{k_0}$  in place of the  $F_{k_0-1}$  and  $F_{k_0}$  defined above. We refer to these intervals  $E_k$  with left endpoint 0 as of type I, and the remaining subsets  $E_k$  as of type II.

LEMMA 2.6. *Let  $p, q$  and  $s$  be admissible exponents. Then for each  $k \in J_{1,\sigma}$  there exist functions  $\psi_l^k$  ( $l = 0, 1, \dots, s$ ) on  $\mathbf{R}_+$  such that  $\text{supp}(\psi_l^k) \subset E_k$ ,  $l = 0, 1, \dots, s$ ,*

$$\frac{1}{E_k} \int_{E_k} \psi_l^k(x) (x - x_0)^j A(x) dx = \delta_{lj}, \quad l, j = 0, 1, \dots, s$$

and

$$|\psi_l^k(x)| \leq \begin{cases} C_{A,p} (2^k \sigma)^{-l-1} |E_k| A(x)^{-1}, & \text{if } E_k \text{ is of type I and } p < 1, \\ C_{A,p} (2^k \sigma)^{-l}, & \text{otherwise} \end{cases}$$

where  $\delta_{lj} = 1$  if  $l = j$  and 0 otherwise.

PROOF. We follow the idea in the proof of [K, Lemma 4.6] and only consider the type I case (the proof for type II intervals is easier and runs similarly). Then  $E_k$  has the following form:

$$E_k = \begin{cases} [0, \sigma], & \text{if } x_0 = 0 \text{ and } k = 0, \\ [0, x_0 + \sigma], & \text{if } 0 < x_0 \leq 1, k_0 \leq 0 \text{ and } k = 0, \\ [0, x_0 + \frac{\sigma}{2}], & \text{if } 0 < x_0 \leq 1, k_0 = 1 \text{ and } k = 0, \\ [0, x_0 - 2^{k_0-2} \sigma], & \text{if } 0 < x_0 \leq 1, k_0 \geq 2 \text{ and } k = k_0 - 1. \end{cases}$$

For  $0 < p \leq 1$  put  $d = [2\alpha + 2]$  if  $p = 1$  and  $d = s$  otherwise. Denote by  $P_i(d; x)$  ( $i = 0, 1, \dots, d$ ) the polynomials of degree  $\leq d$  on the real line determined uniquely by the conditions

$$\frac{1}{E_k} \int_{E_k} P_i(d; x) x^j dx = \delta_{ij}, \quad i, j = 0, 1, \dots, d.$$

Let  $R$  be the right endpoint of the interval  $E_k$  and define for  $l = 0, 1, \dots, d$  and  $k \in J_{1,\sigma}$

$$\psi_l^k(x) := \begin{cases} |E_k| R^{-1} A(x)^{-1} \chi_{E_k}(x) \sum_{i=l}^d \binom{i}{l} x_0^{i-l} R^{-i} P_i\left(d, \frac{x}{R}\right), & p < 1, \\ |E_k| R^{-d-1} x^d A(x)^{-1} \chi_{E_k}(x) P_d\left(d, \frac{x}{R}\right), & p = 1. \end{cases}$$

Using Lemma 1.10 we can verify that  $\psi_l^k$  satisfy the desired conditions. ■

LEMMA 2.7. *Let  $x_0 \in \mathbf{R}_+$  and  $\sigma > 0$ . Then*

$$|B(x_0, \sigma)| \leq C_A 2^{-k} |B(x_0, 2^k \sigma)|, \quad k \in \mathbf{N}_0$$

and

$$E_k \subset \begin{cases} B(x_0, 2^{k+1} \sigma), & k \in J_{1,\sigma}, \\ B(x_0, 2^k \sigma), & k \in J_{2,\sigma}, \end{cases}$$

$$|E_k| \sim |B(x_0, 2^k \sigma)|, \quad k \in \mathbf{N}_0.$$

PROOF. Appealing to (1.2) and Lemma 1.10 we obtain for any  $R > 0$

$$(2.8) \quad |B(x_0, R)| \sim \begin{cases} R^{2\alpha+2}, & x_0 \leq R, R \leq 1, \\ RA(x_0), & x_0 > R, R \leq 1, \\ e^{2\rho(x_0+R)}, & R > 1. \end{cases}$$

The lemma then follows from (2.8) and the definition of  $E_k$ . ■

The following result shows that molecules are generalization of atoms.

LEMMA 2.9. *Every  $(p, q, s)$ -atom  $a$  is a  $(p, q, s, \epsilon)$ -molecule for all  $\epsilon > 0$ , and  $N_q(a) \leq 1$ .*

PROOF. Condition (i) in Definition 2.4 can be verified in the same way as for euclidean spaces. To prove that  $a$  satisfies condition (ii) in Definition 2.4 we can assume  $\text{supp}(a) \subset B(x_0, r)$  where  $r < 1$ . Such an atom satisfies the moment condition

$$\int_0^\infty a(x) x^l A(x) dx = 0, \quad l = 0, 1, \dots, s$$

and then the result follows using (2.8). ■

We now prove the main result of this section: every molecule has an atomic decomposition. From this the molecular characterization of  $\mathbf{h}^p$  will be evident.

THEOREM 2.10. *Let  $M$  be a (local)  $(p, q, s, \epsilon)$ -molecule centred at  $x_0$ . Then  $M \in \mathbf{h}^p$  and*

$$\|M\|_{\mathbf{h}^p} \leq C_{A,p} N_q(M)$$

with  $C_{A,p}$  independent of  $M$ .

PROOF. Consider the sets  $E_k$  ( $k = 0, 1, 2, \dots$ ) where  $\sigma > 0$  is defined as in 2.4(ii), and abbreviate  $\chi_{E_k}$  by  $\chi_k$ . Put  $M_k = M\chi_k$ . For each integer  $k \in J_{1,\sigma}$  define

$$(2.11) \quad P_k(x) := \sum_{j=0}^s m_{kj} \psi_j^k(x) \chi_k(x)$$

where

$$m_{kj} = \frac{1}{|E_k|} \int_0^\infty M_k(x) (x - x_0)^j A(x) dx.$$

Then

$$(2.12) \quad M = \sum_{k=0}^{\infty} M_k = \sum_{k \in J_{1,\sigma}} (M_k - P_k) + \sum_{k \in J_{2,\sigma}} M_k + \sum_{k \in J_{1,\sigma}} P_k.$$

The proof will consist of three parts:

1) To show that each  $M_k - P_k$  ( $k \in J_{1,\sigma}$ ) is a multiple of a local  $(p, 1, s)$ -atom if  $E_k$  is of type I and  $p < 1$  and a local  $(p, q, s)$ -atom otherwise, and that the coefficients sum appropriately.

2) To show that each  $M_k$  ( $k \in J_{2,\sigma}$ ) is a multiple of a local  $(p, q, s)$ -atom and that the coefficients sum appropriately.

3) To show that  $\sum_{k \in J_{1,\sigma}} P_k$  can be written as a sum of local  $(p, 1, s)$ -atoms and  $(p, \infty, s)$ -atoms if  $p < 1$  and a sum of local  $(1, \infty, s)$ -atoms if  $p = 1$ , and that the coefficients sum appropriately.

The theorem will then follow from Theorem 2.3.

Without loss of generality we may assume that  $N_q(M) = 1$ . For each  $k \in \mathbf{N}_0$  applying Lemma 1.10 we obtain from Definition 2.4(ii)

$$\begin{aligned} \|M_k\|_{q,A} &\leq C_{A,p} |B(x_0, 2^{k-1}\sigma)|^{-b} \|M(x) |B(x_0, |x - x_0|)|^b\|_{q,A} \\ &\leq C_{A,p} \left( \frac{|B(x_0, \sigma)|}{|B(x_0, 2^{k-1}\sigma)|} \right)^a |B(x_0, 2^{k-1}\sigma)|^{\frac{1}{q} - \frac{1}{p}} \end{aligned}$$

where  $a = 1 - \frac{1}{p} + \epsilon$  is as in Definition 2.4. Hence by Lemma 2.7

$$(2.13) \quad \|M_k\|_{q,A} \leq C_{A,p} (2^k)^{-a} |B(x_0, 2^k\sigma)|^{\frac{1}{q} - \frac{1}{p}}$$

and similarly

$$(2.14) \quad \|M_k\|_{1,A} \leq C_{A,p} (2^k)^{-a} |B(x_0, 2^k\sigma)|^{1 - \frac{1}{p}}.$$

Let us start with Part 1. Clearly  $\text{supp}(M_k - P_k) \subset B(x_0, 2^k\sigma)$  and, by Lemma 2.6 and (2.11),  $M_k - P_k$  has the right cancellation properties:

$$\int_0^{\infty} (M_k(x) - P_k(x))(x - x_0)^j A(x) dx = |E_k| m_{kj} - |E_k| \sum_{i=0}^s m_{ki} \delta_{ij} = 0.$$

By Lemma 2.6 and (2.11) we have for  $k \in J_{1,\sigma}$

$$|P_k(x)| \leq \begin{cases} C_{A,p} (2^k\sigma)^{-1} A(x)^{-1} \int_{E_k} |M_k(u)| A(u) du, & \text{if } E_k \text{ is of type I and } p < 1, \\ C_{A,p} |E_k|^{-1} \int_{E_k} |M_k(u)| A(u) du, & \text{otherwise.} \end{cases}$$

Consequently for  $k \in J_{1,\sigma}$

$$(2.15) \quad \|P_k\|_{1,A} \leq C_{A,p} \|M_k\|_{1,A}, \quad \text{if } E_k \text{ is of type I and } p < 1$$

and

$$(2.16) \quad \|P_k\|_{q,A} \leq C_{A,p} \|M_k\|_{q,A}, \quad \text{if } E_k \text{ is of type II or } p = 1.$$

Therefore appealing to (2.13)–(2.16) we obtain for each  $k \in J_{1,\sigma}$

$$\|P_k - M_k\|_{1,A} \leq C_{A,p}(2^k)^{-a} |B(x_0, 2^k\sigma)|^{1-\frac{1}{p}}, \quad \text{if } E_k \text{ is of type I and } p < 1$$

and

$$\|P_k - M_k\|_{q,A} \leq C_{A,p}(2^k)^{-a} |B(x_0, 2^k\sigma)|^{\frac{1}{q}-\frac{1}{p}}, \quad \text{if } E_k \text{ is of type II or } p = 1$$

and hence  $a_k^{(1)} := (\lambda_k^{(1)})^{-1}(M_k - P_k)$  is a local  $(p, 1, s)$ -atom if  $E_k$  is of type I and  $p < 1$ , and a local  $(p, q, s)$ -atom otherwise. Here  $\lambda_k^{(1)} = C_{A,p}(2^k)^{-a}$  satisfies  $\sum_{k \in J_{1,\sigma}} |\lambda_k^{(1)}|^p \leq C_{A,p}$ .

For Part 2 we observe that for each  $k \in J_{2,\sigma}$ ,  $M_k$  is supported in  $B(x_0, 2^k\sigma)$  and  $2^k\sigma \geq 1$ . From (2.13) we see that  $a_k^{(2)} := (\lambda_k^{(2)})^{-1}M_k$  is a local  $(p, q, s)$ -atom, and  $\lambda_k^{(2)} = C_{A,p}(2^k)^{-a}$  satisfies  $\sum_{k \in J_{2,\sigma}} |\lambda_k^{(2)}|^p \leq C_{A,p}$ .

Finally we turn to Part 3. Let  $K \in \mathbb{N}_0$  be the integer such that  $2^K\sigma < 1 \leq 2^{K+1}\sigma$  and put  $\tilde{\psi}_l^k = |E_k|^{-1}\psi_l^k\chi_k$ . Then by (2.11)

$$\begin{aligned} \sum_{k \in J_{1,\sigma}} P_k(x) &= \sum_{l=0}^s \sum_{k=0}^K m_{kl} |E_k| \tilde{\psi}_l^k(x) \\ &= \sum_{k=0}^K \sum_{l=0}^s N_l^k \phi_l^k(x) \end{aligned}$$

where  $N_l^k = \sum_{j=0}^k m_{jl} |E_j|$  and

$$\phi_l^k(x) = \begin{cases} \tilde{\psi}_l^k(x) - \tilde{\psi}_l^{k+1}, & k = 0, 1, \dots, K-1, \\ \tilde{\psi}_l^K(x), & k = K. \end{cases}$$

By the definition of  $E_k$  we see that for  $k \in J_{1,\sigma}$  and  $l = 0, 1, \dots, s$

$$(2.17) \quad \begin{aligned} N_l^k &= \sum_{j=0}^k \int_{E_j} M_j(x) (x - x_0)^l A(x) dx \\ &= \begin{cases} \int_{B(x_0, 2^{k_0-2}\sigma)} M(x) (x - x_0)^l A(x) dx \\ \quad + \int_{E_{k_0-1}} M(x) (x - x_0)^l A(x) dx, & \text{if } 0 < x_0 \leq 1, k = k_0 - 1, \\ \int_{B(x_0, 2^k\sigma)} M(x) (x - x_0)^l A(x) dx, & \text{otherwise.} \end{cases} \end{aligned}$$

Applying Definition 2.4(i) and Lemmas 1.10 and 2.7 we obtain for  $j \geq 1$

$$\begin{aligned} \int_{E_j} |M(x)| |x - x_0|^l A(x) dx &\leq (2^j\sigma)^l |E_j| \left\{ \frac{1}{|E_j|} \int_{E_j} |M(x)|^q A(x) dx \right\}^{\frac{1}{q}} \\ &\leq C_{A,p} (2^j\sigma)^l |E_j|^{1-\frac{1}{q}} |B(x_0, 2^j\sigma)|^{-b} \|M(x) B(x_0, |x - x_0|)\|_{q,A}^b \\ &\leq C_{A,p} (2^j\sigma)^l |B(x_0, 2^j\sigma)|^{1-\frac{1}{q}-b} |B(x_0, \sigma)|^a \\ &\leq C_{A,p} (2^j)^{-a} (2^j\sigma)^l |B(x_0, 2^j\sigma)|^{1-\frac{1}{p}} \end{aligned}$$

and

$$\begin{aligned} \int_{E_0} |M(x)| |x - x_0|^l A(x) dx &\leq \sigma^l |E_0|^{1-\frac{1}{q}} \|M\|_{q,A} \\ &\leq C_{A,p} \sigma^l |B(x_0, \sigma)|^{1-\frac{1}{p}}. \end{aligned}$$

Thus for each integer  $j \in \mathbf{N}_0$

$$(2.18) \quad \int_{E_j} |M(x)| |x - x_0|^j A(x) dx \leq C_{A,p} (2^j)^{-a} (2^j \sigma)^j |B(x_0, 2^j \sigma)|^{1-\frac{1}{p}}.$$

Now by (2.17) and (2.18) we apply Definition 2.4(ii) with  $R = 2^k \sigma$  for each  $k \in J_{1,\sigma}$  to obtain

$$|N_l^k| \leq C_{A,p} (2^k)^{-\beta} (2^k \sigma)^l |B(x_0, 2^k \sigma)|^{1-\frac{1}{p}}$$

where  $\beta = \min\{a, s+2-\frac{1}{p}\}$ . By Lemmas 2.6 and 2.7 we observe  $\text{supp}(\phi_l^k) \subset B(x_0, 2^{k+2}\sigma)$  for  $k = 0, 1, \dots, K$  and

$$\int_0^\infty \phi_l^k(x) (x - x_0)^j A(x) dx = 0$$

for  $j = 0, 1, \dots, s$  and  $k = 0, 1, \dots, K-1$ . Notice that  $E_k$  and  $E_{k+1}$  cannot simultaneously be of type I. Therefore using Lemmas 1.10, 2.6 and 2.7 we have for  $k = 0, 1, \dots, K$

$$\sum_{l=0}^s |N_l^k| \|\phi_l^k\|_{1,A} \leq C_{A,p} (2^k)^{-\beta} |B(x_0, 2^k \sigma)|^{1-\frac{1}{p}}$$

if  $p < 1$  and either  $E_k$  or  $E_{k+1}$  is of type I, and

$$\sum_{l=0}^s |N_l^k \phi_l^k(x)| \leq C_{A,p} (2^k)^{-\beta} |B(x_0, 2^k \sigma)|^{-\frac{1}{p}}$$

otherwise. Consequently  $a_k^{(3)} := (\lambda_k^{(3)})^{-1} \sum_{l=0}^s N_l^k \phi_l^k$  is a local  $(p, 1, s)$ -atom if  $p < 1$  and either  $E_k$  or  $E_{k+1}$  is of type I, and a local  $(p, \infty, s)$ -atom otherwise. Here  $\lambda_k^{(3)} := C_{A,p} (2^k)^{-\beta}$  satisfies  $\sum_{k \in J_{1,\sigma}} |\lambda_k^{(3)}|^p \leq C_{A,p}$  since  $\beta = \min\{a, s+2-\frac{1}{p}\} > 0$ . The theorem is therefore proved. ■

We are now in a position to give the following molecular characterization of  $\mathbf{h}^p$ .

**COROLLARY 2.19.** *Let  $f \in \mathcal{S}'_1(\mathbf{R}_+, *(A))$ . Then  $f \in \mathbf{h}^p$  if and only if it has a molecular decomposition:*

$$f = \sum_j M_j$$

where the  $M_j$  are (local)  $(p, q, s, \epsilon)$ -molecules such that

$$\sum_j N_q(M_j)^p < \infty.$$

Moreover if the above decomposition holds then

$$\|f\|_{\mathbf{h}^p} \sim \sum_j N_q(M_j)^p.$$

3. **Fourier multipliers for  $\mathbf{h}^p$ .** We now turn to the Fourier multipliers for  $\mathbf{h}^p$  on  $(\mathbf{R}_+, *(A))$ . After giving estimates for the Fourier transform of functions in  $\mathbf{h}^p$  we introduce the Fourier multipliers for local Hardy spaces and give a necessary condition for a bounded function on the dual space  $\mathbf{R}_+^\wedge$  to be a Fourier multiplier for  $\mathbf{h}^p$ . We then establish a version of Hörmander’s multiplier theorem for  $\mathbf{h}^p$  using atomic and molecular theory.

LEMMA 3.1. *If  $a$  is a local  $(p, q, s)$ -atom then the Fourier transform of  $a$  is holomorphic in the interior of  $F_\delta$ , continuous on  $F_\delta$  and satisfies*

$$|\hat{a}(\lambda)| \leq C_{A,p}(1 + |\lambda|)^{s+1}, \quad \lambda \in F_\delta$$

where  $s = [(2\alpha + 2)(\frac{1}{p} - 1)]$ .

PROOF. Suppose that  $a$  is supported in  $B(x_0, r)$  for some  $x_0 \in \mathbf{R}_+$  and  $r > 0$ . If  $r \geq 1$  then we apply (1.7), Lemma 1.18(i), Definition 2.2 and (2.8) to obtain for  $\lambda = \xi + i\eta \in F_\delta$

$$\begin{aligned} |\hat{a}(\lambda)| &\leq \int_0^\infty |a(x)\varphi_\lambda(x)|A(x) dx \\ &\leq \|a\|_{1,A} \leq |B(x_0, r)|^{1-\frac{1}{p}} \leq C_{A,p} \end{aligned}$$

if  $|\eta| \leq \rho$ , and

$$\begin{aligned} |\hat{a}(\lambda)| &\leq \|a\|_{1,A} e^{(|\eta|-\rho)(x_0+r)} \\ &\leq |B(x_0, r)|^{1-\frac{1}{p}} e^{(|\eta|-\rho)(x_0+r)} \leq C_{A,p} \end{aligned}$$

if  $\eta > \rho$ .

Now assume  $r < 1$ . Using (1.7) and the cancellation property of  $a$  and the Taylor expansion of  $\varphi_\lambda$  about  $x_0$  of order  $s$  we have

$$(3.2) \quad \hat{a}(\lambda) = \frac{1}{(s+1)!} \int_0^\infty a(x)(x-x_0)^{s+1} \varphi_\lambda^{(s+1)}(\xi_x)A(x) dx$$

where  $\xi_x \in B(x_0, r)$ . First consider  $x_0 \leq 2$ . Then by Lemmas 1.17 and 1.10 and (2.8) we obtain for  $\lambda \in F_\delta$

$$\begin{aligned} |\hat{a}(\lambda)| &\leq C_{A,p}r^{s+1} \int_{B(x_0,r)} |a(x)|A(x) dx \\ &\leq C_{A,p}r^{s+1} |B(x_0, r)|^{1-\frac{1}{p}} \leq C_{A,p}r^{s+1+n-\frac{n}{p}} \end{aligned}$$

if  $|\lambda| \leq 1$ , and

$$\begin{aligned} |\hat{a}(\lambda)| &\leq C_{A,p}r^{s+1}(1 + |\lambda|)^{s+1} \left( \int_{\max\{x_0-r,0\}}^{1/|\lambda|} |a(x)|e^{|\eta|x}A(x) dx \right. \\ &\quad \left. + \int_{1/|\lambda|}^{x_0+r} |a(x)|\lambda^{-\alpha-\frac{1}{2}}A(x)^{-\frac{1}{2}}e^{|\eta|x}A(x) dx \right) \\ &\leq C_{A,p}r^{s+1}(1 + |\lambda|)^{s+1} |B(x_0, r)|^{1-\frac{1}{p}} \leq C_{A,p}(1 + |\lambda|)^{s+1} r^{s+1+n-\frac{n}{p}} \end{aligned}$$

if  $|\lambda| > 1$ .

If  $x_0 > 2$  then for  $x \in B(x_0, r)$  we have  $x > x_0 - r > 1$ . Thus appealing to (3.2), Lemmas 1.18(ii) and 1.10 and (2.8) we obtain for  $\lambda \in \bar{F}_\delta$  and  $|\lambda| \leq 2$

$$\begin{aligned} |\hat{a}(\lambda)| &\leq C_{A,p} r^{s+1} \|a\|_{1,A} e^{(\delta-1)\rho(x_0+r)} \\ &\leq C_{A,p} r^{s+1} |B(x_0, r)|^{1-\frac{1}{p}} e^{2\rho(\frac{1}{p}-1)(x_0+r)} \leq C_{A,p} r^{s+2-\frac{1}{p}}, \end{aligned}$$

and to Lemma 1.17 in place of Lemma 1.18(ii) we obtain for  $\lambda \in \bar{F}_\delta$  and  $|\lambda| > 2$

$$\begin{aligned} |\hat{a}(\lambda)| &\leq C_{A,p} r^{s+1} |\lambda|^{s+1-\alpha-\frac{1}{2}} \|a\|_{1,A} e^{(|\eta|-\rho)(x_0+r)} \\ &\leq C_{A,p} r^{s+1} |\lambda|^{s+1} |B(x_0, r)|^{1-\frac{1}{p}} e^{2\rho(\frac{1}{p}-1)(x_0+r)} \leq C_{A,p} r^{s+2-\frac{1}{p}} |\lambda|^{s+1}. \end{aligned}$$

Therefore for  $\lambda \in \bar{F}_\delta$

$$(3.3) \quad |\hat{a}(\lambda)| \leq \begin{cases} C_{A,p}, & r > 1, \\ C_{A,p} (1 + |\lambda|)^{s+1} r^{s+1+n-\frac{n}{p}}, & r \leq 1, x_0 \leq 2, \\ C_{A,p} r^{s+2-\frac{1}{p}} (1 + |\lambda|)^{s+1}, & r \leq 1, x_0 > 2. \end{cases}$$

The result now follows from Lemma 1.12(i), (1.7) and (3.3).  $\blacksquare$

From the definition and Theorem 1.14 we see that the Fourier transform of a tempered distribution is a distribution in  $\mathcal{S}'(F_1)$ . The following result shows that if  $f \in \mathbf{h}^p$  then  $\hat{f}$  is actually analytic on  $\bar{F}_\delta$ .

**THEOREM 3.4.** *Let  $f \in \mathbf{h}^p$ ,  $0 < p \leq 1$ . Then the Fourier transform  $\hat{f}$  is an even function holomorphic in the interior of  $\bar{F}_\delta$  and continuous on  $\bar{F}_\delta$  satisfying*

$$|\hat{f}(\lambda)| \leq C_A \|f\|_{\mathbf{h}^p} (1 + |\lambda|)^{[\frac{n}{p}-n]+1}, \quad \lambda \in \bar{F}_\delta.$$

**PROOF.** By Theorem 2.3 we have a decomposition

$$f = \sum_j \lambda_j a_j$$

where the  $a_j$  are local  $(p, q, s)$ -atoms and

$$\sum_j |\lambda_j|^p \leq C_{A,p} \|f\|_{\mathbf{h}^p}.$$

Since the series converges in  $\mathcal{S}'(\mathbf{R}_+, *(A))$  and the Fourier transform is continuous on  $\mathcal{S}'(\mathbf{R}_+, *(A))$  we have

$$\hat{f} = \sum_j \lambda_j \hat{a}_j.$$

Thus Lemma 3.1 and the fact that

$$\sum_j |\lambda_j| \leq \left( \sum_j |\lambda_j|^p \right)^{1/p} \leq C_{A,p} \|f\|_{\mathbf{h}^p}$$

give the theorem. ■

For a bounded function  $m$  on  $(\mathbf{R}_+, *(A))$  consider the operator  $T_m$  defined by

$$(3.5) \quad (T_m f)^\wedge(\lambda) = m(\lambda)\hat{f}(\lambda).$$

By Theorem 3.4,  $T_m$  is a well-defined continuous operator from  $\mathbf{h}^p$  ( $0 < p \leq 1$ ) to  $\mathcal{S}'(\mathbf{R}_+, *(A))$  and by Theorem 1.8,  $T_m$  is bounded on  $L^2(\mathbf{R}_+, Adx)$  whenever  $m$  is a bounded function. A bounded function  $m$  is said to be a *Fourier multiplier* for  $\mathbf{h}^p$  if the operator  $T_m$  takes  $\mathbf{h}^p$  continuously into  $\mathbf{h}^p$ .

The following theorem shows that some holomorphy of the function  $m$  is necessary for  $T_m$  to be bounded on  $\mathbf{h}^p$ . This new phenomenon, different from the euclidean case, arises from the exponential growth of the hypergroups.

LEMMA 3.6. *Let  $0 < p \leq 1$ . Then every Fourier multiplier  $m$  for  $\mathbf{h}^p$  extends to an even function holomorphic in the interior of the strip  $F_\delta$  and continuous on  $F_\delta$ .*

PROOF. Choose  $f(x) = h_1(x)$  where  $h_t(x)$  is the heat kernel (see [AT]). Now applying the semigroup property of the heat kernel:

$$h_{t_1} * h_{t_2} = h_{t_1+t_2}$$

and Definition 2.1 we see that  $f \in \mathbf{h}^p$  for  $0 < p \leq 1$ . Observe that  $\hat{f}(\lambda) = e^{-(\lambda^2+\rho^2)}$  is holomorphic and does not vanish. If  $m$  is a Fourier multiplier for  $\mathbf{h}^p$  then  $T_m f \in \mathbf{h}^p$  and, by (3.5),  $m(\lambda) = \frac{(T_m f)^\wedge(\lambda)}{\hat{f}(\lambda)}$ . The lemma now follows readily from Theorem 3.4. ■

We now establish a version of the Hörmander-Mihlin multiplier theorem for  $\mathbf{h}^p$  Fourier multipliers on  $(\mathbf{R}_+, *(A))$ , but first we begin with some definitions. The notation  $K$  is reserved for the kernel obtained as the Fourier transform of  $m$  in the distributional sense. Then  $T_m f = f * K$ . Choose an even  $C^\infty$ -function  $\psi$  on  $\mathbf{R}$  such that  $\psi(x) = 1$  for  $|x| \leq \frac{1}{2}$  and  $\psi(x) = 0$  for  $|x| \geq 1$ , and fix once and for all a kernel decomposition  $K = K^0 + K^\infty$  where  $K^0 = K\psi$  and  $K^\infty = K(1 - \psi)$ .

DEFINITION 3.7. For a positive integer  $N$  we say that a bounded function  $m$  satisfies a Hörmander condition of order  $N$  (and denote this by  $m \in \mathcal{M}(2, N)$ ) if  $m$  extends to an even analytic function inside  $F_\delta$  and the derivatives  $m^{(i)}$  extend continuously to the whole of  $F_\delta$  and satisfy

$$\sup_{\lambda \in F_\delta} (1 + |\lambda|)^i |m^{(i)}(\lambda)| < \infty, \quad i = 0, 1, \dots, N.$$

For  $m \in \mathcal{M}(2, N)$  set  $\|m\|_{\mathcal{M}(2, N)} := \max_{0 \leq i \leq N} \sup_{\lambda \in F_\delta} (1 + |\lambda|)^i |m^{(i)}(\lambda)|$ .

Let  $\phi$  be an even nonnegative  $C^\infty$ -function supported in  $\{x \in \mathbf{R} : \frac{1}{2} < |x| < 2\}$  and satisfying  $\sum_{j=-\infty}^\infty \phi(2^{-j}x) = 1$  for  $x \neq 0$ . Put  $\phi_j(x) = \phi(2^{-j}x)$  for  $j = 1, 2, \dots$  and  $\phi_0(x) = 1 - \sum_{j=1}^\infty \phi(2^{-j}x)$ . For  $m \in \mathcal{M}(2, N)$  we fix once and for all a dyadic decomposition  $m(\lambda) = \sum_{k=0}^\infty m_k(\lambda)$  where  $m_k(\lambda) = m(\lambda)\phi_k(\lambda)$ . Thus the corresponding decomposition for the kernel  $K$  is  $K(x) = \sum_{k=0}^\infty K_k(x)$  where  $\hat{K}_k(\lambda) = m_k(\lambda)$ . Throughout



the remainder of the paper we shall always assume that  $m$  is in fact rapidly decreasing (i.e.  $m \in \mathcal{S}(F_0)$ , the usual Schwartz space) though none of our estimates will depend upon the actual rate of decrease. It suffices to flatten  $m$  or, equivalently, to regularize  $K$  in the standard way. Thus by Theorem 1.14,  $K \in \mathcal{S}_2(\mathbf{R}_+, *(A))$ .

The proof of the following lemma is similar to that of [An, Proposition 5].

LEMMA 3.8. *Suppose that  $m \in \mathcal{M}(2, N)$  with  $N = [\frac{2\alpha+2}{p} - \alpha - 1] + 1$  and  $0 < p \leq 1$ . Then  $K^\infty \in L^1(\mathbf{R}_+, Adx)$  and*

$$\|K^\infty\|_{1,A} \leq C_A \|m\|_{M(2,N)}.$$

LEMMA 3.9. *For any  $0 < R \leq 1$  we have*

$$\int_0^R y |K^0(y)| A(y) dy \leq C_A \|m\|_{M(2,N)} R$$

and

$$\int_R^1 y^{-1} |K^0(y)| A(y) dy \leq C_A \|m\|_{M(2,N)} R^{-1}.$$

PROOF. We only give the proof of the first inequality (the second can be handled similarly). Let  $k_0$  be the positive integer such that  $1 \leq 2^{k_0} R < 2$ . Using the dyadic decomposition of  $m$  we observe

$$\begin{aligned} \int_0^R y |K^0(y)| A(y) dy &\leq \sum_{k=0}^{\infty} \int_0^R |K_k^0(y)| y A(y) dy \\ &= \sum_{k=0}^{k_0} \int_0^R |K_k^0(y)| y A(y) dy + \sum_{k=k_0+1}^{\infty} \int_0^{2^{-k}} |K_k^0(y)| y A(y) dy \\ &\quad + \sum_{k=k_0+1}^{\infty} \int_{2^{-k}}^R |K_k^0(y)| y A(y) dy \\ &:= \sigma_1 + \sigma_2 + \sigma_3 \end{aligned}$$

where  $K_k^0 = \psi K$ . Now applying Theorems 1.8 and 1.9 and properties of the classical Fourier transform we have

$$\begin{aligned} \sigma_1 &\leq C_A R^{\alpha+2} \sum_{k=0}^{k_0} \left\{ \int_0^\infty |K_k^0(y)|^2 A(y) dy \right\}^{1/2} \\ &\leq C_A R^{\alpha+2} \sum_{k=0}^{k_0} \left\{ \int_0^\infty |m_k(\lambda)|^2 (1+\lambda)^{2\alpha+1} d\lambda \right\}^{1/2} \\ &\leq C_A R^{\frac{3}{2}} \sum_{k=0}^{k_0} \left\{ \int_0^\infty |m_k(\lambda)|^2 d\lambda \right\}^{1/2} \\ &\leq C_A R^{\frac{3}{2}} \|m\|_{M(2,N)} \sum_{k=0}^{k_0} 2^{\frac{k}{2}} \leq C_A \|m\|_{M(2,N)} R \end{aligned}$$

and

$$\begin{aligned} \sigma_2 &\leq C_A(2^{-k})^{\alpha+2} \sum_{k=k_0}^{\infty} \left\{ \int_0^{\infty} |K_k^0(y)|^2 A(y) dy \right\}^{1/2} \\ &\leq C_A(2^{-k})^{\alpha+2} \sum_{k=k_0}^{\infty} \left\{ \int_0^{\infty} |m_k(\lambda)|^2 (1 + \lambda)^{2\alpha+1} d\lambda \right\}^{1/2} \\ &\leq C_A(2^{-k})^{\frac{3}{2}} \sum_{k=k_0}^{\infty} \left\{ \int_0^{\infty} |m_k(\lambda)|^2 d\lambda \right\}^{1/2} \\ &\leq C_A \|m\|_{M(2,N)} R. \end{aligned}$$

To estimate  $\sigma_3$  we introduce smooth cut-off functions as in [An]. Let  $\omega^0$  be an even  $C^\infty$ -function on  $\mathbf{R}$  such that  $\omega^0(x) = 1$  for  $|x| \leq \frac{1}{4}$  and  $\omega^0(x) = 0$  for  $|x| \geq \frac{1}{2}$ , and set  $\omega_j^0(x) = \omega^0(2^j x)$  for each  $j \in \mathbf{N}_0$ . Then  $\omega_j^0(x) = 1$  for  $|x| \leq 2^{-j-2}$ ,  $\omega_j^0(x) = 0$  for  $|x| \geq 2^{-j-1}$  and  $|\frac{d^i}{dx^i} \omega_j^0(x)| \leq C_i 2^{ij}$ ,  $i = 0, 1, 2, \dots$ . Denote by  $l$  the inverse classical Fourier transform of  $m$ . For a dyadic decomposition of  $m$  let  $l(u) = \sum_{k=0}^{\infty} l_k(u)$  be the corresponding decomposition where  $F_0 l_k(\lambda) = m_k(2^{-k} \lambda)$ . Put  $l_{kj}^0 = (1 - \omega_j^0) l_k$  and let  $K_{kj}^0 = A^{-1}(l_{kj}^0)$  and  $m_{kj}^0 = F_0(l_{kj}^0)$ . Then  $l_k - l_{kj}^0$  is an even  $C^\infty$ -function supported in  $[-2^{-j-1}, 2^{-j-1}]$ , and hence using the property of the Abel transform (see [T, Théorème 6.4]) we see that  $K_k - K_{kj}^0 = A^{-1}(l_k - l_{kj}^0)$  is also supported in  $[-2^{-j-1}, 2^{-j-1}]$ . Consequently

$$(3.10) \quad K_k(x) = K_{kj}^0(x), \quad x > 2^{-j-1}.$$

We now apply (3.10), Theorems 1.8 and 1.9 and the properties of the classical Fourier transform to obtain

$$\begin{aligned} \sigma_3 &\leq C_A \sum_{k=k_0+1}^{\infty} \sum_{j=k_0-1}^k \int_{2^{-j-1}}^{2^{-j}} |K_k^0(y)| y A(y) dy \\ &\leq C_A \sum_{k=k_0+1}^{\infty} \sum_{j=k_0-1}^k (2^{-j})^{\alpha+2} \left\{ \int_{2^{-j-1}}^{2^{-j}} |K_k(y)|^2 A(y) dy \right\}^{1/2} \\ &= C_A \sum_{k=k_0+1}^{\infty} \sum_{j=k_0-1}^k (2^{-j})^{\alpha+2} \left\{ \int_{2^{-j-1}}^{2^{-j}} |K_{kj}^0(y)|^2 A(y) dy \right\}^{1/2} \\ &\leq C_A \sum_{k=k_0+1}^{\infty} \sum_{j=k_0-1}^k (2^{-j})^{\alpha+2} \left\{ \int_{2^{-j-1}}^{2^{-j}} |m_{kj}^0(\lambda)|^2 (1 + \lambda)^{2\alpha+1} d\lambda \right\}^{1/2}. \end{aligned}$$

Put  $\Omega_j^0 = 1 - \omega_j^0$ . Then  $m_{kj}^0 = F_0(\Omega_j^0 l_k)$  and hence

$$\sigma_3 \leq C_A \sum_{k=k_0+1}^{\infty} \sum_{j=k_0-1}^k (2^{-j})^{\alpha+2} \left\{ \int_{2^{-j-1}}^{2^{-j}} |F_0(\Omega_j^0 l_k)(\lambda)|^2 (1 + \lambda)^{\alpha+\frac{1}{2}} d\lambda \right\}^{1/2}.$$

Arguing as in [An, Lemma 15] we have for  $\beta_1 < \beta_2$

$$\left( \sum_{j=0}^k (2^{-\beta_2 j} \|\Omega_j^0 l_k\|_{H_2^{\beta_1}})^2 \right)^{\frac{1}{2}} \leq C_A 2^{k(\frac{1}{2} + \beta_1 - \beta_2)} \|m\|_{M(2,N)}$$

where  $H_2^\beta$  is the usual Sobolev space. Therefore by substituting  $\beta_1 = \alpha + \frac{1}{2}$  and  $\beta_2 = \alpha + 2$  we obtain

$$\sigma_3 \leq C_A \|m\|_{M(2,N)} \sum_{k=k_0+1}^{\infty} 2^{-k} \leq C_A \|m\|_{M(2,N)} R$$

and this completes the proof of the lemma.  $\blacksquare$

Let  $\psi$  be the function defining  $K^0$ , and for  $R > 0$  and  $l \in \mathbf{N}_0$  put  $Q_{R,l}(x) = \psi_R(x)(x-x_0)$  where  $\psi_R(x) = \psi(\frac{x-x_0}{R})$ .

LEMMA 3.11. For any  $0 < R \leq 1$  and  $k, l \in \mathbf{N}_0$  we have

- (i)  $|\frac{\partial^k}{\partial x^k} T_y Q_{R,l}(x)| \leq C_{A,k,l} R^{l-k}$ ,  $|x-x_0| < R$ ,  $y \in \mathbf{R}_+$ ,
  - (ii)  $|\frac{\partial}{\partial y} \frac{\partial^k}{\partial x^k} T_y Q_{R,l}(x)| \leq C_{A,k,l} R^{l-k-1}$ ,  $|x-x_0| < R$ ,  $0 < y \leq 1$ , and
  - (iii)  $|\frac{\partial^k}{\partial x^k} T_y Q_{R,l}(x)| \leq C_{A,k,l} R^{l-k+1} y^{-1}$ ,  $|x-x_0| < R$ ,  $R \leq y \leq 1$ .
- Here  $T_y$  is the generalized translation defined by (1.12).

PROOF. By the definition of  $Q_{R,l}$  we see that

$$|Q_{R,l}^{(k)}(x)| \leq C_{A,k,l} R^{l-k}, \quad k \in \mathbf{N}_0.$$

Thus the lemma can be proved in the same way as in [BX2, Lemma 3.15] using Theorem 1.9 and Lemmas 1.10 and 1.17.

LEMMA 3.12. Suppose that  $m \in \mathcal{M}(2, N)$  with  $N = [\frac{n}{p} - \frac{n}{2}] + 1$  and  $0 < p \leq 1$ , and  $a$  is a local  $(p, q, s)$ -atom supported in  $B(x_0, r)$  with  $r < 1$ . If  $T_m a$  is a local  $(p, q, s, \epsilon)$ -molecule then for any  $\sigma \leq R \leq 1$

$$\left| \int_{B(x_0, R)} T_m a(x) (x-x_0)^l A(x) dx \right| \leq C_{A,l} R^l |B(x_0, R)|^{1-\frac{1}{p}} \left( \frac{\sigma}{R} \right)^\beta$$

for  $l = 0, 1, \dots, s$ , where  $\sigma$  is the positive number defined by  $|B(x_0, \sigma)|^{\frac{1}{q}-\frac{1}{p}} = \|T_m a\|_{q,A}$  and  $\beta = \min\{1 - \frac{1}{p} + \epsilon, s + 2 - \frac{1}{p}\}$  as in Definition 2.4.

PROOF. We first observe that an application of the Hörmander's multiplier theorem for  $L^q(\mathbf{R}_+, Adx)$  ( $q > 1$ ) gives

$$(3.13) \quad \|T_m a\|_{q,A} \leq C_{A,q} \|a\|_{q,A}.$$

By Definition 2.2 we see that (3.13) implies that

$$(3.14) \quad |B(x_0, r)| \leq C_{A,q} |B(x_0, \sigma)|.$$

If  $R < 2r$  then we apply the Cauchy-Schwarz inequality, (3.14) and (2.8) to obtain

$$\begin{aligned} \left| \int_{B(x_0, R)} T_m a(x) (x-x_0)^l A(x) dx \right| &\leq C_A R^l \|T_m a\|_{q,A} |B(x_0, R)|^{1-\frac{1}{q}} \\ &= C_A R^l |B(x_0, r)|^{1-\frac{1}{q}} |B(x_0, \sigma)|^{\frac{1}{q}-\frac{1}{p}} \\ &= C_A R^l |B(x_0, R)|^{1-\frac{1}{p}} \left( \frac{|B(x_0, R)|}{|B(x_0, \sigma)|} \right)^{\frac{1}{p}-\frac{1}{q}} \\ &\leq C_{A,p} R^l |B(x_0, R)|^{1-\frac{1}{p}} \left( \frac{\sigma}{R} \right)^\beta. \end{aligned}$$

We now assume that  $R \geq 2r$  and write

$$\begin{aligned} \int_{B(x_0,R)} T_m a(x)(x-x_0)^j A(x) dx &= \int_{B(x_0,R)} T_m a(x)\psi_R(x)(x-x_0)^j A(x) dx \\ &\quad + \int_{B(x_0,R)} T_m a(x)(1-\psi_R(x))(x-x_0)^j A(x) dx \\ &:= I_R^{(1)} + I_R^{(2)}. \end{aligned}$$

Let  $E_R := \{x \in \mathbf{R}_+ : \frac{R}{2} < |x-x_0| < R\}$ . As  $T_m a$  is a  $(p, q, s, \epsilon)$ -molecule by assumption we argue similarly as in showing (2.18) to obtain

$$|I_R^{(2)}| \leq C_{A,p} R^l |B(x_0, R)|^{-\epsilon} |B(x_0, \sigma)|^{1-\frac{1}{p}+\epsilon}.$$

Hence by (2.8)

$$|I_R^{(2)}| \leq C_{A,p} R^l |B(x_0, R)|^{1-\frac{1}{p}} \left(\frac{\sigma}{R}\right)^\beta.$$

It remains to estimate  $I_R^{(1)}$  for  $R \geq 2r$ . Using the property of the generalized translation  $T_y$  and the decomposition of the kernel  $K$  we observe

$$\begin{aligned} I_R^{(1)} &= \int_{B(x_0,R)} T_m a(x)\psi_R(x)(x-x_0)^j A(x) dx \\ &= \int_0^\infty K(y) \left[ \int_0^\infty a(x) T_y \mathcal{Q}_{R,l}(x) A(x) dx \right] A(y) dy \\ &= \int_0^\infty K^0(y) \left[ \int_0^\infty a(x) T_y \mathcal{Q}_{R,l}(x) A(x) dx \right] A(y) dy \\ &\quad + \int_0^\infty K^\infty(y) \left[ \int_0^\infty a(x) T_y \mathcal{Q}_{R,l}(x) A(x) dx \right] A(y) dy \\ &= \int_0^R K^0(y) \left[ \int_0^\infty a(x) T_y \mathcal{Q}_{R,l}(x) A(x) dx \right] A(y) dy \\ &\quad + \int_R^1 K^0(y) \left[ \int_0^\infty a(x) T_y \mathcal{Q}_{R,l}(x) A(x) dx \right] A(y) dy \\ &\quad + \int_0^\infty K^\infty(y) \left[ \int_0^\infty a(x) T_y \mathcal{Q}_{R,l}(x) A(x) dx \right] A(y) dy \\ &:= I_R^{(1,1)} + I_R^{(1,2)} + I_R^{(1,3)} \end{aligned}$$

where  $\mathcal{Q}_{R,l} = \psi_R(x)(x-x_0)^l$  is as in Lemma 3.11. Note that  $\psi_R(x) = 1$  for  $|x-x_0| \leq \frac{1}{2}$ . Hence using properties of the generalized translation we have

$$T_0 \mathcal{Q}_{R,l}(x) = \mathcal{Q}_{R,l}(x) = (x-x_0)^l, \quad |x-x_0| < r, \quad R \geq 2r.$$

Now using the cancellation properties and the Taylor expansion of  $F(x, y) = T_y \mathcal{Q}_{R,l}(x)$  we obtain

$$\begin{aligned} \int_0^\infty a(x) T_y \mathcal{Q}_{R,l}(x) A(x) dx &= \int_{B(x_0,r)} a(x) [T_y \mathcal{Q}_{R,l}(x) - \mathcal{Q}_{R,l}(x)] A(x) dx \\ &= \frac{y}{(s+1)!} \int_0^1 \int_0^1 (1-v)^s \left[ \int_{B(x_0,r)} a(x)(x-x_0)^{s+1} \right. \\ &\quad \left. \times \left( \frac{\partial^{s+1}}{\partial \eta \partial \xi^{s+1}} T_\eta \mathcal{Q}_{R,l}(\xi) \right)_{\substack{\xi=x_0+v(x-x_0) \\ \eta=uy}} A(x) dx \right] du dv. \end{aligned}$$

Thus applying Lemma 3.11(ii) and Definition 2.2 we have for  $0 < y \leq 1$

$$\left| \int_0^\infty a(x) T_y Q_{R,t}(x) A(x) dx \right| \leq C_{A,p} r^{s+1} R^{l-s-2} y |B(x_0, r)|^{1-\frac{1}{p}}$$

and hence by Lemma 3.9, (2.8) and (3.14)

$$|I_R^{(1,1)}| \leq C_{A,t} R^l |B(x_0, R)|^{1-\frac{1}{p}} \left(\frac{\sigma}{R}\right)^\beta.$$

Similarly using the Taylor expansion of  $T_y Q_{R,t}$  about  $x_0$  and Lemma 3.11(i), (iii) we obtain for  $R \leq y \leq 1$

$$\left| \int_0^\infty a(x) T_y Q_{R,t}(x) A(x) dx \right| \leq C_{A,p} r^{s+1} R^{l-s} y^{-1} |B(x_0, r)|^{1-\frac{1}{p}}$$

and for any  $y \in \mathbf{R}_+$

$$\left| \int_0^\infty a(x) T_y Q_{R,t}(x) A(x) dx \right| \leq C_{A,p} r^{s+1} R^{l-s-1} |B(x_0, r)|^{1-\frac{1}{p}}.$$

Therefore applying Lemmas 3.8 and 3.9, (2.8) and (3.14) we obtain

$$|I_R^{(1,i)}| \leq C_{A,t} R^l |B(x_0, R)|^{1-\frac{1}{p}} \left(\frac{\sigma}{R}\right)^\beta, \quad i = 2, 3$$

and this completes the proof of the lemma.  $\blacksquare$

We also need the following estimates concerning  $m$  and its corresponding kernel  $K$ .

LEMMA 3.15. *Suppose that  $m \in \mathcal{M}(2, N)$  with  $N = [\frac{2\alpha+2}{p} - \alpha - 1] + 1$ ,  $\alpha \geq 0$  and  $0 < p \leq 1$ . Then for  $0 < t \leq 1$  and  $|x - y| \geq 2$*

$$|T_x(K * h_t)(y)| \leq C_A \mathcal{M}(2, N) |x - y|^{-N} A(x)^{-\frac{1}{2}} A(y)^{-\frac{1}{2}} e^{-\delta\rho|x-y|}$$

where  $\delta = \frac{2}{p} - 1$ ,  $h_t$  is the heat kernel and  $K$  is the kernel corresponding to  $m$ .

PROOF. We follow [An] and choose  $\omega \in C^\infty(\mathbf{R})$  such that  $\omega(x) = 0$  for  $x \leq \frac{1}{2}$  and  $\omega(x) = 1$  for  $x \geq 1$ . For any fixed  $x, y \in \mathbf{R}_+$  with  $|x - y| \geq 2$  write

$$\omega_{|x-y|}(u) := \omega(|x - y| + u) \omega(|x - y| - u).$$

Then  $\omega_{|x-y|}$  is an even  $C^\infty$ -function on  $\mathbf{R}$  satisfying  $\omega_{|x-y|}(u) = 1$  for  $|u| < |x - y| - 1$  and  $\omega_{|x-y|}(u) = 0$  for  $|u| \geq |x - y| - \frac{1}{2}$ . Writing  $l := \hat{A}(K * h_t)$  and  $\tilde{m} := F_0 l$  we see that  $\tilde{m}(\lambda) = m(\lambda) e^{-t(\lambda^2 + \rho^2)}$ . Put  $l_{|x-y|} := l(1 - \omega_{|x-y|})$ ,  $K_{|x-y|} := \hat{A}^{-1} l_{|x-y|}$  and  $m_{|x-y|} := F_0 l_{|x-y|}$ . Now  $l - l_{|x-y|}$  is supported in  $[0, |x - y| - \frac{1}{2}]$ . Hence by [T, Théorème 6.4] we have  $\text{supp}(K * h_t - K_{|x-y|}) \subset [0, |x - y| - \frac{1}{2}]$  which implies that

$$K * h_t(u) = K_{|x-y|}(u), \quad u > |x - y| - \frac{1}{2}.$$

Thus by (1.12),  $T_x(K * h_t)(y) = T_x K_{|x-y|,t}(y)$ , and by [BH, Theorem 2.2.36] and [BX1, (2.17) and (2.18)]

$$(3.16) \quad T_x(K * h_t)(y) = \int_0^\infty m_{|x-y|,t}(\lambda) \varphi_\lambda(x) \varphi_\lambda(y) |c(\lambda)|^{-2} d\lambda.$$

We claim now that for any  $L$  with  $0 \leq L < N - \frac{1}{2}$  and  $|x - y| \geq 2$

$$(3.17) \quad \left\{ \int_0^\infty |m_{|x-y|}(\lambda)(1 + \lambda)^L|^2 d\lambda \right\}^{1/2} \leq C_A \|m\|_{M(2,N)} |x - y|^{-N} e^{\delta\rho|x-y|}.$$

In fact by interpolation we can restrict ourselves to the case when  $l \in \mathbf{N}_0$ . In view of the properties of the classical Fourier transform and the analyticity of  $m$  we have

$$\begin{aligned} & \left\{ \int_0^\infty |m_{|x-y|}(\lambda)(1 + \lambda)^L|^2 d\lambda \right\}^{1/2} \\ & \leq C_A \sum_{i=0}^L \left\{ \int_0^\infty |l_{|x-y|}^{(i)}(u)|^2 du \right\}^{1/2} \\ & \leq C_A \sum_{i=0}^L \sum_{j=0}^i t^{-i+j} \left\{ \int_{|x-y|^{-1}}^\infty |l^{(j)}(u)|^2 du \right\}^{1/2} \\ & \leq C_A |x - y|^{-N} e^{-\delta\rho|x-y|} \sum_{i=0}^L \sum_{j=0}^i \left\{ \int_0^\infty |u^N e^{\delta\rho u} l^{(j)}(u)|^2 du \right\}^{1/2} \\ & \leq C_A |x - y|^{-N} e^{-\delta\rho|x-y|} \sum_{i=0}^L \sum_{j=0}^i \left\{ \int_0^\infty \left| \frac{d^N}{d\lambda^N} ((\lambda + i\delta\rho)^j m(\lambda + i\delta\rho)) \right|^2 d\lambda \right\}^{1/2} \\ & \leq C_A |x - y|^{-N} e^{-\delta\rho|x-y|} \sum_{i=0}^L \left\{ \int_0^\infty |(\lambda + i\delta\rho)^{L-i} m^{(N-i)}(\lambda + i\delta\rho)|^2 d\lambda \right\}^{1/2} \\ & \leq C_A |x - y|^{-N} e^{-\delta\rho|x-y|} \|m\|_{M(2,N)}. \end{aligned}$$

The lemma now follows from (3.16) and (3.17) using Theorem 1.8, Lemmas 1.17 and 1.10 and a straightforward calculation. ■

Let  $\omega$  be the function as in the proof of Lemma 3.15. For any integer  $j > 1$  we define an even  $C^\infty$ -function  $\omega_j$  by

$$\omega_j(u) = \omega(2(u + j - 1))\omega(2(-u + j - 1)),$$

and denote by  $l$  the Abel transform of  $K$ . Then by Theorem 1.14,  $m = F(K) = F_0(l)$ . Put  $l_j = (1 - \omega_j)l$ ,  $m_j = F_0(l_j)$  and  $K_j = A^{-1}(l_j)$ . Since  $l - l_j$  is supported in  $[-j + \frac{5}{4}, j - \frac{5}{4}]$ , by the properties of the Abel transform in [T, Théorème 6.4] we see that  $K - K_j$  is also supported in  $[-j + \frac{5}{4}, j - \frac{5}{4}]$  and hence

$$(3.18) \quad K(x) = K_j(x), \quad \text{if } x > j - \frac{5}{4}.$$

LEMMA 3.19. *Suppose that  $m \in M(2, N)$  with  $N = [\frac{2\alpha+2}{p} - \frac{n}{2}] + 1$ ,  $\alpha \geq 0$  and  $0 < p \leq 1$ . Then for any  $L$  with  $0 \leq L < N - \frac{1}{2}$*

$$\left\{ \int_0^\infty |m_j(\lambda)(1 + \lambda)^L|^2 d\lambda \right\}^{1/2} \leq C_A \|m\|_{M(2,N)} j^{-N} e^{\delta\rho j}, \quad j = 2, 3, \dots$$

PROOF. The proof of the lemma is similar to that of (3.17). ■

For  $m \in \mathcal{M}(2, N)$  fix a dyadic decomposition  $m = \sum_{k=0}^{\infty} m_k$  and the corresponding decompositions  $K = \sum_{k=0}^{\infty} K_k$  and  $l = \sum_{k=0}^{\infty} l_k$  where  $F(K_k) = F_0(l_k) = m_k$ . Choose an even  $C^\infty$ -function  $\omega^0$  such that

$$\omega^0 = \begin{cases} 1, & |x| \leq \frac{1}{4}, \\ 0, & |x| \geq \frac{1}{2}. \end{cases}$$

For any positive integer  $j$  and  $r > 0$  put  $l_{kj} := (1 - \tilde{\omega}_j)l_k$ ,  $K_{kj} := A^{-1}(l_{kj})$  and  $m_{kj} := F_0(l_{kj})$  where  $\tilde{\omega}_j(x) := \omega^0(\frac{x}{2^j r})$ . Observe that  $l_k - l_{kj}$  is supported in  $\{u : |u| \leq 2^j r\}$ . Using the properties of the Abel transform in [T] we have

$$(3.20) \quad K_k(x) = K_{kj}(x), \quad \text{if } x > 2^{j-2}r.$$

LEMMA 3.21. *Suppose that  $m \in \mathcal{M}(2, N)$  with  $N = [\frac{2\alpha+2}{p} - \alpha - 1] + 1$ ,  $\alpha \geq 0$  and  $0 < p \leq 1$ . Given  $j \in \mathbf{N}_0$  and  $r > 0$  such that  $2^j r \leq 1$  we have for any nonnegative numbers  $L_1$  and  $L_2$  with  $L_2 \leq N$*

$$\left\{ \int_0^\infty |m_{kj}(\lambda)(1 + \lambda)^{L_1}|^2 d\lambda \right\}^{\frac{1}{2}} \leq \begin{cases} C_{A,m}(2^j r)^{-L_1} 2^{\frac{k}{2}}, & 2^{j+k}r < 1, \\ C_{A,m}(2^j r)^{-L_2} 2^{k(L_1 - L_2 + \frac{1}{2})}, & \text{otherwise.} \end{cases}$$

where  $C_{A,m} = C_A \|m\|_{\mathcal{M}(2,N)}$ .

PROOF. We only consider the case when  $L_1 \leq L_2$  and  $2^{j+k}r \geq 1$  (the proof of the other cases is similar), and by interpolation we can restrict ourselves to integers  $L_1$  and  $L_2$ . Applying properties of the classical Fourier transform and the classical Plancherel theorem we obtain

$$\begin{aligned} \left\{ \int_0^\infty |m_{kj}(\lambda)(1 + \lambda)^{L_1}|^2 d\lambda \right\}^{1/2} &\leq C_A \sum_{i=0}^{L_1} \left\{ \int_0^\infty |l_{kj}^{(i)}(u)|^2 du \right\}^{1/2} \\ &\leq C_A \sum_{i=0}^{L_1} \left\{ \int_{2^{j-3}r}^{2^{j-2}r} |l_{kj}^{(i)}(u)|^2 du \right\}^{1/2} \\ &\quad + C_A \sum_{i=0}^{L_1} \left\{ \int_{2^{j-2}r}^\infty |l_k^{(i)}(u)|^2 du \right\}^{1/2} \\ &:= I_1 + I_2. \end{aligned}$$

Using the definition of  $l_{kj}$  and  $m_{kj}$  and properties of the classical Fourier transform we have

$$\begin{aligned} I_1 &\leq C_A \sum_{i=0}^{L_1} \sum_{n=0}^i (2^j r)^{n-i} \left\{ \int_{2^{j-3}r}^{2^{j-2}r} |l_k^{(n)}(u)|^2 du \right\}^{1/2} \\ &\leq C_A (2^j r)^{-L_2} \sum_{i=0}^{L_1} \sum_{n=0}^i \left\{ \int_{2^{j-3}r}^{2^{j-2}r} |l_k^{(n)}(u) u^{n-i+L_2}|^2 du \right\}^{1/2} \\ &\leq C_A (2^j r)^{-L_2} \sum_{i=0}^{L_1} \sum_{n=0}^i \left\{ \int_0^\infty \left| \frac{d^{L_2+n-i}}{d\lambda^{L_2+n-i}} (\lambda^n m_k(\lambda)) \right|^2 d\lambda \right\}^{1/2} \\ &\leq C_A (2^j r)^{-L_2} \sum_{i=0}^{L_1} \left\{ \int_0^\infty |\lambda^{L-i} m_k^{(N-i)}(\lambda)|^2 d\lambda \right\}^{1/2}. \end{aligned}$$

Now recall that

$$m_k(\lambda) = m(\lambda)\phi(2^{-k}\lambda) \text{ for } k = 1, 2, \dots \text{ and } m_0(\lambda) = m(\lambda)\left(1 - \sum_{k=1}^{\infty} \phi(2^{-k}\lambda)\right).$$

Hence

$$|m_k^{(L_2-i)}(\lambda)| \leq C_A 2^{k(i-L_2)}$$

and

$$I_1 \leq C_A \|m\|_{M(2,N)} (2^j r)^{-L_2} 2^{k(L_1-L_2+\frac{1}{2})}.$$

Similarly

$$I_2 \leq C_A \|m\|_{M(2,N)} (2^j r)^{-L_2} 2^{k(L_1-L_2+\frac{1}{2})},$$

and this completes the proof of the lemma. ■

We now give a version of Hörmander’s multiplier theorem for local Hardy spaces.

**THEOREM 3.22.** *Suppose that  $m \in \mathcal{M}(2, N)$  with  $N = [\frac{2\alpha+2}{p} - \alpha - 1] + 1$ ,  $\alpha \geq 0$  and  $0 < p \leq 1$ . Then  $m$  is a Fourier multiplier for  $\mathbf{h}^p$ .*

**PROOF.** By Definition 2.1 and Theorem 2.3 we are reduced to showing that for any (local)  $(p, \infty, s)$ -atom  $a$

$$(3.23) \quad \|T_m a\|_{\mathbf{h}^p} \leq C_{A,p}$$

where the constant  $C_{A,p}$  is independent of  $a$ .

Suppose that  $a$  is supported in  $B(x_0, r)$  with  $x_0 \in \mathbf{R}_+$  and  $r > 0$ . If  $r > 1$  then we write

$$\begin{aligned} \|H_0^+(T_m a)\|_{\mathbf{h}^p}^p &= \int_0^{x_0+r+2} |H_0^+(T_m a)(x)|^p A(x) dx \\ &\quad + \int_{x_0+r+2}^{\infty} |H_0^+(T_m a)(x)|^p A(x) dx \\ &:= I_1 + I_2. \end{aligned}$$

Note that both  $H_0^+$  and  $T_m$  are  $L^2$ -bounded (see [BX2]) and any  $(p, \infty, s)$ -atom must be a  $(p, q, s)$ -atom for all  $q > p$ ,  $1 \leq q < \infty$ . Applying Hölder’s inequality, Lemma 1.10 and (2.8) then gives

$$\begin{aligned} I_1 &\leq C_{A,p} \|H_0^+(T_m a)\|_{2,A}^p e^{2\rho(x_0+r+2)(1-\frac{q}{2})} \\ &\leq C_{A,p} \|a\|_{2,A}^p e^{2\rho(x_0+r+2)(1-\frac{q}{2})} \\ &\leq C_{A,p} |B(x_0, r)|^{\frac{q}{2}-1} e^{2\rho(x_0+r+2)(1-\frac{q}{2})} \leq C_{A,p}. \end{aligned}$$

To estimate  $I_2$  we first observe

$$T_m a * h_t = a * h_t * K(x) = \int_0^{\infty} a(y) T_x(h_t * K)(y) A(y) dy.$$



For  $y \in B(x_0, r)$  and  $x > x_0 + r + 2$  we have  $x - y > 2$ . Hence by Lemmas 3.15 and 1.10 we have for  $x > x_0 + r + 2$  and  $0 < t \leq 1$

$$\begin{aligned} |T_m a * h_t(x)| &\leq C_{A,p} A(x)^{-\frac{1}{2}} e^{-\delta \rho x} \int_0^\infty |a(y)| (x-y)^{-N} e^{\delta \rho y} A(y)^{\frac{1}{2}} dy \\ &\leq C_{A,p} A(x)^{-\frac{1}{p}} (x-x_0-r)^{-N} \|a\|_{2,A} e^{\delta \rho(x_0+r)} \\ &\leq C_{A,p} A(x)^{-\frac{1}{p}} (x-x_0-r)^{-N} e^{(\delta+1-\frac{2}{p})\rho(x_0+r)} \\ &= C_{A,p} A(x)^{-\frac{1}{p}} (x-x_0-r)^{-N}. \end{aligned}$$

Consequently by the definition of  $H_0^+$

$$I_2 \leq C_{A,p} \int_{x_0+r+2}^\infty (x-x_0-r)^{-Np} dx \leq C_{A,p}$$

and (3.23) follows for  $r > 1$ .

We now assume  $r \leq 1$ . As before let  $\psi$  be an even  $C^\infty$ -function such that  $\psi(x) = 1$  if  $|x| \leq \frac{1}{2}$  and  $\psi(x) = 0$  if  $|x| \geq 1$ , and  $\phi$  an even nonnegative  $C^\infty$ -function supported in  $\{x \in \mathbf{R} : \frac{1}{2} < |x| < 2\}$  and satisfying  $\sum_{j=-\infty}^\infty \phi(2^{-j}x) = 1$  if  $x \neq 0$ . Write

$$\begin{aligned} T_m a(x) &:= \sum_{j=-\infty}^\infty T_m a(x) \phi_j(x) \tilde{\psi}(x) + T_m a(x) (1 - \tilde{\psi}(x)) \\ &:= (T_m a)_1(x) + (T_m a)_2(x) \end{aligned}$$

where  $\phi_j(x) = \phi(\frac{x-x_0}{2^j r})$  and  $\tilde{\psi}(x) = \psi(\frac{x-x_0}{4})$ . We first prove that  $(T_m a)_2$  has an atomic decomposition and then that  $(T_m a)_1$  is a  $(p, 2, s, \epsilon)$ -molecule.

For each  $j = 2, 3, \dots$  let  $Q_j = \{x \in \mathbf{R}_+ : j < |x-x_0| \leq j+1\}$ . Note that  $(T_m a)_2(x) = 0$  if  $|x-x_0| \leq 2$ . Hence

$$(T_m a)_2(x) = \sum_{j=2}^\infty (T_m a)_2(x) \chi_{Q_j}(x) := \sum_{j=2}^\infty b_j(x).$$

Using (3.18), (1.11) and (1.12) together with the cancellation property of an atom we observe for  $x \in Q_j$

$$\begin{aligned} T_m a(x) &= \int_0^\infty a(y) T_x K(y) A(y) dy \\ &= \int_0^\infty a(y) T_x K_j(y) A(y) dy \\ &= \int_0^\infty a(y) (y-x_0)^s \int_0^1 (1-u)^{s-1} F_{j,y,u,s}(x) du A(y) dy \\ &= \int_0^1 (1-u)^{s-1} \int_0^\infty a(y) (y-x_0)^s F_{j,y,u,s}(x) A(y) dy du \end{aligned}$$

if  $s > 0$ , and

$$T_m a(x) = \int_0^\infty a(y) F_{j,y,u,s}(x) A(y) dy$$

if  $s = 0$ , where

$$F_{j,y,u,s}(x) = \begin{cases} (T_x K_j)^{(s)}(x_0 + u(y - x_0)) - (T_x K_j)^{(s)}(x_0), & s > 0, \\ T_x K_j(y) - T_x K_j(x_0), & s = 0. \end{cases}$$

By [BH, Theorem 2.2.36] and [BX1, (2.17) and (2.18)]

$$(T_x K_j)^{(k)}(y) = \int_0^\infty m_j(\lambda) \varphi_\lambda(x) \varphi_\lambda^{(k)}(y) |c(\lambda)|^{-2} d\lambda, \quad k \in \mathbf{N}_0.$$

Thus

$$\hat{F}_{j,y,x_0,u}(\lambda) = \begin{cases} m_j(\lambda) [\varphi_\lambda^{(s)}(x_0 + u(y - x_0)) - \varphi_\lambda^{(s)}(x_0)], & s > 0, \\ m_j(\lambda) [\varphi_\lambda(y) - \varphi_\lambda(x_0)], & s = 0. \end{cases}$$

We only consider  $s > 0$  (the case  $s = 0$  can be handled similarly). Applying Theorems 1.8 and 1.9 gives

$$\begin{aligned} & \|F_{j,y,u,s}\|_{2,A} \\ &= \left\{ \int_0^\infty \left| m_j(\lambda) \left( \varphi_\lambda^{(s)}(x_0 + u(y - x_0)) - \varphi_\lambda^{(s)}(x_0) \right) \right|^2 \sigma(d\lambda) \right\}^{\frac{1}{2}} \\ &\leq C_A \sum_{u|y-x_0|2^k \leq 1} \left\{ \int_0^\infty \left| \tilde{m}_{jk}(\lambda) \left( \varphi_\lambda^{(s)}(x_0 + u(y - x_0)) - \varphi_\lambda^{(s)}(x_0) \right) \right|^2 \sigma(d\lambda) \right\}^{\frac{1}{2}} \\ &\quad + C_A \sum_{u|y-x_0|2^k > 1} \left\{ \int_0^\infty \left| \tilde{m}_{jk}(\lambda) \left( \varphi_\lambda^{(s)}(x_0 + u(y - x_0)) - \varphi_\lambda^{(s)}(x_0) \right) \right|^2 \sigma(d\lambda) \right\}^{\frac{1}{2}} \\ &:= \sum_1 + \sum_2 \end{aligned}$$

where  $\tilde{m}_{jk}(\lambda) = m_j(\lambda)\phi_k(\lambda)$ ,  $\phi_0(\lambda) = 1 - \sum_{i=1}^\infty \phi(2^{-i}\lambda)$  and  $\phi_k(\lambda) = \phi(2^{-k}\lambda)$ . For  $x_0 \leq 1$  choose  $\gamma > 0$  such that  $\gamma < \min(N - \frac{2\alpha+2}{p} + \alpha + 1, N - s - \alpha - 1, s - \frac{2\alpha+2}{p} + 2\alpha + 3)$ , and in addition  $\gamma < N - s - \alpha - 2$  if  $N - s - \alpha - 2 > 0$ . Then we use Lagrange’s mean-value theorem, Theorem 1.9 and Lemmas 1.17 and 3.19 to obtain

$$\begin{aligned} \sum_1 &\leq C_A u |y - y_0| \sum_{u|y-x_0|2^k \leq 1} \left\{ \int_0^\infty |\tilde{m}_{jk}(\lambda) \varphi_\lambda^{(s+1)}(\xi)|^2 |c(\lambda)|^{-2} d\lambda \right\}^{\frac{1}{2}} \\ &\leq C_A u |y - y_0| \left\{ \int_0^\infty |m_j(\lambda) \phi_0(\lambda) (1 + \lambda)^{s+1}|^2 |c(\lambda)|^{-2} d\lambda \right\}^{\frac{1}{2}} \\ &\quad + C_A u |y - y_0| \sum_{0 < u|y-x_0|2^k \leq 1} \left\{ \int_0^\infty |m_j(\lambda) \phi_k(\lambda) (1 + \lambda)^{s+1}|^2 |c(\lambda)|^{-2} d\lambda \right\}^{\frac{1}{2}} \\ &\leq C_A u |y - y_0| \left\{ \int_0^\infty |m_j(\lambda)|^2 d\lambda \right\}^{\frac{1}{2}} \\ &\quad + C_A u |y - y_0| \sum_{0 < u|y-x_0|2^k \leq 1} 2^{k(s+\alpha+3-N+\gamma)} \left\{ \int_0^\infty |m_j(\lambda) (1 + \lambda)^{N-\frac{1}{2}-\gamma}|^2 d\lambda \right\}^{\frac{1}{2}} \\ &\leq \begin{cases} C_A j^{-N} e^{-\delta \rho j} \|m\| M_{(2,N)} u |y - x_0|, & N - s - \alpha - 2 > 0, \\ C_A j^{-N} e^{-\delta \rho j} \|m\| M_{(2,N)} (u|y - x_0|)^{N-s-\alpha-1-\gamma}, & N - s - \alpha - 2 < 0, \\ C_A j^{-N} e^{-\delta \rho j} \|m\| M_{(2,N)} (u|y - x_0|)^{1-\gamma}, & N - s - \alpha - 2 = 0 \end{cases} \end{aligned}$$

and

$$\begin{aligned} \sum_2 &\leq C_A \sum_{|y-x_0|2^k > 1} \left\{ \int_0^\infty |\tilde{m}_{jk}(\lambda)(1+\lambda)^s|^2 (1+\lambda)^{2\alpha+1} d\lambda \right\}^{1/2} \\ &\leq C_A \sum_{0 < |y-x_0|2^k > 1} 2^{k(s+\alpha+1-N+\gamma)} \left\{ \int_0^\infty |m_j(\lambda)\phi_k(\lambda)(1+\lambda)^{N-\frac{1}{2}-\gamma}|^2 d\lambda \right\}^{1/2} \\ &\leq C_A j^{-N} e^{-\delta\rho j} \|m\|_{M(2,N)} (|y-x_0|)^{N-s-\alpha-1-\gamma}. \end{aligned}$$

Thus for  $x_0 \leq 1$  we have by the definition of an atom and (2.8)

$$\begin{aligned} \|b_j\|_{2,A} &\leq \int_0^1 (1-u)^{s-1} \left[ \int_0^\infty a(y)(y-x_0)^s \|F_{j,y,u,s}\|_{2,A} A(y) dy \right] du \\ &\leq C_A j^{-N} e^{-\delta\rho j} \|m\|_{M(2,N)} \\ &\leq j^{-N} \|m\|_{M(2,N)} |B(x_0, j+1)|^{\frac{1}{2}-\frac{1}{p}}. \end{aligned}$$

If  $x_0 > 1$  then we argue similarly to obtain

$$\sum_1 \leq \begin{cases} C_A j^{-N} e^{-\delta\rho j} \|m\|_{M(2,N)} A(x_0)^{-\frac{1}{2}} |y-x_0|, & N-s-\frac{3}{2} > 0, \\ C_A j^{-N} e^{-\delta\rho j} \|m\|_{M(2,N)} A(x_0)^{-\frac{1}{2}} (|y-x_0|)^{N-s-\frac{1}{2}-\gamma_1}, & N-s-\frac{3}{2} < 0, \\ C_A j^{-N} e^{-\delta\rho j} \|m\|_{M(2,N)} A(x_0)^{-\frac{1}{2}} (|y-x_0|)^{1-\gamma_1}, & N-s-\frac{3}{2} = 0 \end{cases}$$

and

$$\sum_2 \leq C_A j^{-N} e^{-\delta\rho j} \|m\|_{M(2,N)} A(x_0)^{-\frac{1}{2}} (|y-x_0|)^{N-s-\frac{1}{2}-\gamma_1}$$

where  $\gamma_1 > 0$  is chosen so that  $\gamma_1 < \min(N - \frac{1}{p} + \frac{1}{2}, N - s - \frac{1}{2}, s - \frac{1}{p} + 2)$ , and in addition  $\gamma_1 < N - s - \frac{3}{2}$  if  $N - s - \frac{3}{2} > 0$ . Hence by (2.8)

$$\begin{aligned} \|b_j\|_{2,A} &\leq C_A j^{-N} e^{-\delta\rho j} \|m\|_{M(2,N)} A(x_0)^{\frac{1}{2}-\frac{1}{p}} \\ &\leq j^{-N} \|m\|_{M(2,N)} |B(x_0, j+1)|^{\frac{1}{2}-\frac{1}{p}}. \end{aligned}$$

Observe that  $\text{supp}(b_j) \subset B(x_0, j+1)$ . Therefore  $a_j := C_A^{-1} \|m\|_{M(2,N)}^{-1} j^N b_j$  is a (local)  $(p, 2, s)$ -atom for each  $j = 2, 3, \dots$  and

$$\begin{aligned} (3.24) \quad \|(T_m a)_2\|_{\mathbf{h}^p} &\leq C_{A,p} \|m\|_{M(2,N)} \left\{ \sum_{j=2}^\infty j^{-Np} \|a_j\|_{\mathbf{h}^p}^p \right\}^{1/2} \\ &\leq C_{A,p} \|m\|_{M(2,N)}. \end{aligned}$$

It remains to show that

$$(3.25) \quad \|(T_m a)_1\|_{\mathbf{h}^p} \leq C_{A,p} \|m\|_{M(2,N)}$$

and then (3.23) for  $r \leq 1$  will follow from (3.24) and (3.25). Let  $j_0$  be the unique non-negative integer such that  $1 \leq 2^{j_0}r < 2$ . Then

$$\begin{aligned} (T_m a)_1(x) &= \sum_{j=-\infty}^{\infty} T_m a(x) \tilde{\psi}(x) \phi_j(x) = \sum_{j=-\infty}^{j_0+2} T_m a(x) \tilde{\psi}(x) \phi_j(x) \\ &:= \sum_{j=-\infty}^{j_0+2} \tilde{b}_j(x). \end{aligned}$$

Fix a dyadic decomposition  $m = \sum_{k=0}^{\infty} m_k$  and the corresponding decomposition  $K = \sum_{k=0}^{\infty} K_k$  as before. By (3.20), (1.11) and (1.12) and using the moment condition of an atom we observe for  $j = 2, 3, \dots, j_0 + 2$  and  $x \in \text{supp}(\phi_j)$

$$\begin{aligned} \tilde{b}_j(x) &= \tilde{\psi}(x) \phi_j(x) \sum_{k=0}^{\infty} \int_0^{\infty} a(y) T_x K_k(y) A(y) dy \\ &= \tilde{\psi}(x) \phi_j(x) \sum_{k=0}^{\infty} \int_0^{\infty} a(y) T_x K_{kj}(y) A(y) dy \\ &= \tilde{\psi}(x) \phi_j(x) \int_0^1 (1-u)^{s-1} \left\{ \int_0^{\infty} a(y) (y-x_0)^s G_{k,j,y,u,s}(x) A(y) dy \right\} du \end{aligned}$$

where

$$G_{k,j,y,u,s}(x) = \begin{cases} (T_x K_{kj})^{(s)}(x_0 + u(y-x_0)) - (T_x K_{kj})^{(s)}(x_0), & s > 0, \\ T_x K_{kj}(y) - T_x K_{kj}(x_0), & s = 0. \end{cases}$$

Note that  $\hat{G}_{k,j,y,u,s}(\lambda) = m_{kj}(\lambda) (\varphi_{\lambda}^{(s)}(x_0 + u(y-x_0)) - \varphi_{\lambda}^{(s)}(x_0))$  if  $s > 0$  and  $\hat{G}_{k,j,y,u,s}(\lambda) = m_{kj}(\lambda) (\varphi_{\lambda}(y) - \varphi_{\lambda}(x_0))$  if  $s = 0$ . Hence by Theorems 1.8 and 1.9

$$\begin{aligned} &\sum_{k=0}^{\infty} \|\tilde{\psi} \phi_j G_{k,j,y,u,s}\|_{2,A} \\ &\leq \|G_{k,j,y,u,s}\|_{2,A} \\ &= \sum_{k=0}^{\infty} \left\{ \int_0^{\infty} \left| m_{kj}(\lambda) (\varphi_{\lambda}^{(s)}(x_0 + u(y-x_0)) - \varphi_{\lambda}^{(s)}(x_0)) \right|^2 \sigma(d\lambda) \right\}^{\frac{1}{2}} \\ &\leq C_A \sum_{|y-x_0|2^k \leq 1} \left\{ \int_0^{\infty} \left| m_{kj}(\lambda) (\varphi_{\lambda}^{(s)}(x_0 + u(y-x_0)) - \varphi_{\lambda}^{(s)}(x_0)) \right|^2 \sigma(d\lambda) \right\}^{\frac{1}{2}} \\ &\quad + C_A \sum_{|y-x_0|2^k > 1} \left\{ \int_0^{\infty} \left| m_{kj}(\lambda) (\varphi_{\lambda}^{(s)}(x_0 + u(y-x_0)) - \varphi_{\lambda}^{(s)}(x_0)) \right|^2 \sigma(d\lambda) \right\}^{\frac{1}{2}} \\ &:= J_1 + J_2. \end{aligned}$$

Assume that  $x_0 \leq 2r$ . Then by Lagrange's mean-value theorem, Theorem 1.9 and Lemmas 1.17 and 3.21 with  $L_1 = s + \alpha + \frac{3}{2}$  and  $L_2 = N$  we have

$$\begin{aligned}
J_1 &\leq C_A u |y - x_0| \sum_{u|y-x_0|2^k \leq 1} \left\{ \int_0^\infty |m_{kj}(\lambda) \varphi_\lambda^{(s+1)}(\xi)|^2 |c(\lambda)|^{-2} d\lambda \right\}^{\frac{1}{2}} \\
&\leq C_A u |y - y_0| \sum_{u|y-x_0|2^k \leq 1} \left\{ \int_0^\infty |m_{kj}(\lambda)(1 + \lambda)^{s+\alpha+\frac{3}{2}}|^2 d\lambda \right\}^{\frac{1}{2}} \\
&\leq C_A u |y - x_0| \sum_{\substack{u|y-x_0| \leq 1 \\ 2^{k+j}r \leq 1}} (2^j r)^{-s-\alpha-\frac{3}{2}} 2^{\frac{k}{2}} \\
&\quad + C_A u |y - x_0| \sum_{\substack{u|y-x_0| \leq 1 \\ 2^{k+j}r > 1}} (2^j r)^{-N} 2^{k(s+\alpha+2-N)} \\
&\leq C_A u |y - x_0| (2^j r)^{-s-\alpha-2}
\end{aligned}$$

if  $N - s - \alpha - 2 > 0$ , and

$$\begin{aligned}
J_1 &\leq u |y - x_0| (2^j r)^{-N} \sum_{u|y-x_0| \leq 1} 2^{k(s+\alpha+2-N)} \\
&\leq C_A (u |y - x_0|)^{N-s-\alpha-1} (2^j r)^{-N}
\end{aligned}$$

if  $N - s - \alpha - 2 < 0$ . For the particular case when  $N - s - \alpha - 2 = 0$  we write

$$\begin{aligned}
J_1 &= C_A \sum_{\substack{u|y-x_0| \leq 1 \\ 2^{k+j}r \leq 1}} \left\{ \int_0^\infty |m_{kj}(\lambda) (\varphi_\lambda^{(s)}(x_0 + u(y-x_0)) - \varphi_\lambda^{(s)}(x_0))|^2 \sigma(d\lambda) \right\}^{\frac{1}{2}} \\
&\quad + C_A \sum_{\substack{u|y-x_0| \leq 1 \\ 1 < 2^{k+j}r \leq 2^j}} \left\{ \int_0^\infty |m_{kj}(\lambda) (\varphi_\lambda^{(s)}(x_0 + u(y-x_0)) - \varphi_\lambda^{(s)}(x_0))|^2 \sigma(d\lambda) \right\}^{\frac{1}{2}} \\
&\quad + C_A \sum_{\substack{u|y-x_0| \leq 1 \\ 2^{k+j}r > 2^j}} \left\{ \int_0^\infty |m_{kj}(\lambda) (\varphi_\lambda^{(s)}(x_0 + u(y-x_0)) - \varphi_\lambda^{(s)}(x_0))|^2 \sigma(d\lambda) \right\}^{\frac{1}{2}} \\
&:= J_{1,1} + J_{1,2} + J_{1,3}.
\end{aligned}$$

Now using Theorem 1.9 and Lemmas 1.17 and 3.21 (with  $L_1 = s + \alpha + \frac{3}{2}$  and  $L_2 = N$  for  $J_{1,1}$  and  $L_1 = s + \alpha + \frac{1}{2}$  for  $J_{1,3}$ ) we obtain

$$\begin{aligned}
J_{1,1} &\leq C_A u |y - x_0| \sum_{\substack{u|y-x_0| \leq 1 \\ 2^{k+j}r \leq 1}} \left\{ \int_0^\infty |m_{kj}(\lambda)(1 + \lambda)^{s+\alpha+\frac{3}{2}}|^2 d\lambda \right\}^{\frac{1}{2}} \\
&\leq C_A u |y - x_0| (2^j r)^{-N}
\end{aligned}$$

and

$$\begin{aligned}
J_{1,3} &\leq C_A \sum_{\substack{u|y-x_0| \leq 1 \\ 2^{k+j}r > 2^j}} \left\{ \int_0^\infty |m_{kj}(\lambda)(1 + \lambda)^{s+\alpha+\frac{1}{2}}|^2 d\lambda \right\}^{\frac{1}{2}} \\
&\leq C_A r (2^j r)^{-N}.
\end{aligned}$$

For  $J_{1,2}$  we choose  $\beta > 0$  sufficiently small such that  $N - \frac{2\alpha+2}{p} + \alpha + 1 - \beta > 0$  and apply Lemma 3.21 with  $L_1 = s + \alpha + \frac{3}{2}$  and  $L_2 = N - \beta$  to obtain

$$\begin{aligned} J_{1,2} &\leq C_A u |y - x_0| (2^j r)^{-N+\beta} \sum_{\substack{u|y-x_0| \leq 1 \\ 1 < 2^{kj} r \leq 2^j}} 2^{k\beta} \\ &\leq C_A u |y - x_0| (2^j r)^{-N+\beta} r^{-\beta}. \end{aligned}$$

Hence

$$J_1 \leq \begin{cases} C_A r (2^j r)^{-s-\alpha-2}, & N - s - \alpha - 2 > 0, \\ C_A r^{1-\beta} (2^j r)^{-N+\beta}, & N - s - \alpha - 2 = 0, \\ C_A r^{N-s-\alpha-1} (2^j r)^{-N}, & N - s - \alpha - 2 < 0. \end{cases}$$

Similarly applying Theorem 1.9 and Lemmas 1.17 and 3.21 (with  $L_1 = s + \alpha + \frac{1}{2}$  and  $L_2 = N$ ) we have

$$\begin{aligned} J_2 &\leq C_A \sum_{u|y-x_0| 2^k > 1} \left\{ \int_0^\infty |m_{kj}(\lambda)(1+\lambda)^s|^2 \sigma(d\lambda) \right\}^{\frac{1}{2}} \\ &\leq r^{N-s-\alpha-1} (2^j r)^{-N}. \end{aligned}$$

Therefore by Definition 2.2, (2.8) and Lemma 1.10

$$(3.26) \quad \|\tilde{b}_j\|_{2,A} \leq C_{A,p} 2^{-j\mu_1} |B(x_0, 2^{j+1}r)|^{\frac{1}{2}-\frac{1}{p}}$$

where

$$\mu_1 = \begin{cases} s - \frac{2\alpha+2}{p} + 2\alpha + 3, & \text{if } N - s - \alpha - 2 > 0, \\ N - \frac{2\alpha+2}{p} + \alpha + 1, & \text{if } N - s - \alpha - 2 < 0, \\ N - \frac{2\alpha+2}{p} + \alpha + 1 - \beta, & \text{if } N - s - \alpha - 2 = 0 \end{cases}$$

and  $j = 2, 3, \dots, j_0 + 2$ . If  $x_0 > 2r$  then a similar argument gives

$$(3.27) \quad \|\tilde{b}_j\|_{2,A} \leq C_{A,p} 2^{-j\mu_2} |B(x_0, 2^{j+1}r)|^{\frac{1}{2}-\frac{1}{p}}$$

where

$$\mu_2 = \begin{cases} s - \frac{1}{p} + 2, & \text{if } N - s - \frac{3}{2} > 0, \\ N - \frac{1}{p} + \frac{1}{2}, & \text{if } N - s - \frac{3}{2} < 0, \\ N - \frac{1}{p} + \frac{1}{2} - \beta, & \text{if } N - s - \frac{3}{2} = 0 \end{cases}$$

and  $j = 2, 3, \dots, j_0 + 2$ .

We now prove that  $(T_m a)_1$  is a (local)  $(p, 2, s, \epsilon)$ -molecule with  $\epsilon$  satisfying  $\max\{\frac{s}{2\alpha+2}, \frac{1}{p} - 1\} < \epsilon < \min\{\frac{s+1}{2\alpha+2}, \frac{N}{2\alpha+2} - \frac{1}{2} - \frac{\beta}{2\alpha+2}\}$ . First by Definition 2.2 and the fact that  $T_m$  is  $L^2$ -bounded we observe

$$\|(T_m a)_1\|_{2,A} \leq \|T_m a\|_{2,A} \leq \|a\|_{2,A} \leq C_{A,p} |B(x_0, r)|^{\frac{1}{2}-\frac{1}{p}}.$$

Put  $\mu = \mu_1$  if  $x_0 \leq 2r$  and  $\mu = \mu_2$  otherwise. By (3.26), (3.27) and (2.8) we have with  $a = 1 - \frac{1}{p} + \epsilon$  and  $b = \frac{1}{2} + \epsilon$  as in Definition 2.4.

$$\begin{aligned}
& \left\| (T_m a)_1(x) |B(x_0, |x - x_0|)|^b \right\|_{2,A}^{1-\frac{a}{b}} \\
& \leq C_{A,p} \left\| \sum_{j=-\infty}^1 \tilde{b}_j \right\|_{2,A}^{1-\frac{a}{b}} |B(x_0, r)|^{b-a} \\
& \quad + C_{A,p} \sum_{j=2}^{j_0+2} \left\| \tilde{b}_j \right\|_{2,A}^{1-\frac{a}{b}} |B(x_0, 2^{j+1}r)|^{b-a} \\
& \leq C_{A,p} |B(x_0, r)|^{\frac{1}{p} - \frac{1}{2}} \|T_m a\|_{2,A}^{1-\frac{a}{b}} \\
& \quad + C_{A,p} \sum_{j=2}^{j_0+2} 2^{-j\mu(1-\frac{a}{b})} |B(x_0, 2^{j+1}r)|^{\frac{a}{b}(\frac{1}{p} - \frac{1}{2})} \\
& \leq C_{A,p} \sum_{j=1}^{j_0+2} 2^{-j\mu(1-\frac{a}{b})} |B(x_0, 2^{j+1}r)|^{\frac{a}{b}(\frac{1}{p} - \frac{1}{2})} \\
& \leq \begin{cases} C_{A,p} r^{\frac{a}{b}(\frac{2\alpha+2}{p} - \alpha - 1)} \sum_{j=1}^{j_0+2} (2^{-j})^{(1-\frac{a}{b})(\mu - (2\alpha+2)a)}, & x_0 \leq 2r, \\ C_{A,p} (rA(x_0))^{\frac{a}{b}(\frac{1}{p} - \frac{1}{2})} \sum_{j=1}^{j_0+2} (2^{-j})^{(1-\frac{a}{b})(\mu - (2\alpha+2)a)}, & x_0 > 2r \end{cases} \\
& \leq C_{A,p} |B(x_0, r)|^{\frac{a}{b}(\frac{1}{p} - \frac{1}{2})}.
\end{aligned}$$

Consequently

$$(3.28) \quad N_2((T_m a)_1) \leq C_{A,p}.$$

For any  $0 < R \leq 1$  and  $l = 0, 1, \dots, s$  we have

$$\int_{B(x_0, R)} (T_m a)_1(x) (x - x_0)^l A(x) dx = \int_{B(x_0, R)} T_m a(x) (x - x_0)^l A(x) dx.$$

Thus by Lemma 3.12 and Definition 2.4,  $(T_m a)_1$  is a  $(p, 2, s, \epsilon)$ -molecule. The estimate (3.25) now follows from (3.28) and Theorem 2.10, and this completes the proof of the theorem.  $\blacksquare$

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