Powers in a class of $A$-strict standard episturmian words

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Introduction

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- A natural generalization of Sturmian words to an arbitrary finite alphabet.
- Sturmian words are exactly the aperiodic episturmian words over a 2-letter alphabet.
Aim

- Explicitly determine all integer powers occurring in episturmian words.
- This has been done for Sturmian words by Damanik & Lenz (2003).
- We do this for a restricted class of episturmian words.
Terminology and notation

Let $A$ denote a finite alphabet and let $u = x_1 x_2 \cdots x_m$ where each $x_i \in A$. 
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Powers in a class of $\mathcal{A}$-strict standard episturmian words – p.4/17
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For $0 \leq j \leq m - 1$, the $j$-th conjugate of $u$ is the word

$$C_j(u) := x_{j+1}x_{j+2} \cdots x_mx_1x_2 \cdots x_j$$

and we define

$$C(u) := \{C_j(u) : 0 \leq j \leq |u| - 1\},$$

the **conjugacy class** of $u$.
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- A factor of $x$ is a finite string of consecutive letters in $x$.
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- A factor $w$ of $x$ is
  \[
  \begin{cases}
    \text{right special} & \text{if } \{wa, wb\} \\
    \text{left} & \text{if } \{aw, bw\}
  \end{cases}
  \]
  are factors of $x$ for some $a, b \in A$, $a \neq b$. 
An infinite word $t$ is *episturmian* if:

- $\Omega(t)$ is closed under reversal, and
- $t$ has at most one right special factor of each length.
Episturmian words

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An episturmian word is *standard* if all of its left special factors are prefixes of it.
Let $t$ be a standard episturmian word over $A$ and let

$$u_1 = \varepsilon, \ u_2, \ u_3, \ u_4, \ldots$$

be the sequence of its palindromic prefixes.
Standard episturmian words

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There exists an infinite word $\Delta(t) = x_1 x_2 x_3 \ldots (x_i \in \mathcal{A})$ such that

$$u_{n+1} = (u_n x_n)^{(+)}, \quad n \in \mathbb{N}^+.$$  

Note: $w^{(+)}$ is the shortest palindrome of which $w$ is a prefix.
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\( \Delta(t) \) is called the *directive word* of \( t \).
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- $\Delta(t)$ is called the \textit{directive word} of $t$.

- $t = \lim_{n \to \infty} u_n$
A class of strict standard episturmian words

Take \( \mathcal{A} = \mathcal{A}_k := \{a_1, a_2, \ldots, a_k\} \).
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$t$ is \textit{k-strict} if each $a_i \in A_k$ appears infinitely often in $\Delta(t)$. 
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For any standard episturmian word $t$,

$$
\Delta(t) = a_1^{d_1} a_2^{d_2} \cdots a_k^{d_k} a_1^{d_{k+1}} a_2^{d_{k+2}} \cdots a_k^{d_{2k}} a_1^{d_{2k+1}} \cdots,
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where each $d_i \geq 0$. 

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where each $d_i \geq 0$.

We restrict our attention to the case when all $d_i > 0$.

Let $s$ be the $k$-strict standard episturmian word with directive word:

$$\Delta(s) = a_1^{d_1} a_2^{d_2} \cdots a_k^{d_k} a_1^{d_{k+1}} a_2^{d_{k+2}} \cdots a_k^{d_{2k}} a_1^{d_{2k+1}} \cdots, \quad d_i > 0.$$
Example

- Let $\alpha \in (0, 1)$ be irrational with $\alpha = [0; 1 + d_1, d_2, d_3, \ldots]$.
- The characteristic Sturmian word $c_\alpha$ over $\{a, b\}$ has directive word

$$\Delta(c_\alpha) = a^{d_1} b^{d_2} a^{d_3} b^{d_4} a^{d_5} \ldots.$$
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- $c_\alpha = \lim_{n \to \infty} s_n$, where $(s_n)_{n \geq -1}$ is defined by

$$s_{-1} = b, \quad s_0 = a, \quad s_n = s_{n-1}^{d_n}s_{n-2}, \quad n \geq 1.$$
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Powers in a class of $A$-strict standard episturmian words – p.9/17
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$\Delta(s)$ resembles $\Delta(c_\alpha)$.

One can prove that $s = \lim_{n \to \infty} s_n$ where the sequence $\{s_n\}_{n \geq 1-k}$ is defined by

$$s_1 = a_2, \quad s_2 = a_3, \quad \ldots, \quad s_{-1} = a_k, \quad s_0 = a_1,$$

$$s_n = \begin{cases} s_{n-1}^{d_n} s_{n-2}^{d_{n-1}} \cdots s_0^{d_1} a_{n+1}, & 1 \leq n \leq k - 1, \\ s_n = \begin{cases} s_{n-1}^{d_n} s_{n-2}^{d_{n-1}} \cdots s_{n-k+1}^{d_{n-k+2}} s_{n-k}, & n \geq k \end{cases} \end{cases}.$$
Let $p \geq 2$ be an integer.

A finite word $w$ has a $p$-th power in $s$ if

$$w^p = \underbrace{ww \cdots w}_{p}$$

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Powers

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is a factor of $s$.

**Key tools** in our analysis of powers occurring in $s$:

- canonical decompositions of $s$ in terms of its building blocks $s_n$;
- a generalization of singular words.
The set of factors of length $|s_n|$ in $c_\alpha$ is given by

$$\{\text{all conjugates of } s_n\} \cup \{w_n\}$$

where $w_n$ is called the $n$-th singular factor of $c_\alpha$.

[Wen and Wen (1994), Melançon (1999), Cao and Wen (2003)]
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Singular $n$-words of the $i$-th kind

The set of factors of $s$ of length $|s_n|$ can be partitioned into $k$ sets.
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**Singular $n$-words of the $i$-th kind**

The set of factors of $s$ of length $|s_n|$ can be partitioned into $k$ sets.

That is:

$$\Omega_{|s_n|}(s) = \mathcal{C}(s_n) \cup \Omega^1_n \cup \cdots \cup \Omega^{k-1}_n.$$
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\Omega_{|s_n|}(s) = C(s_n) \cup \Omega_n^1 \cup \cdots \cup \Omega_n^{k-1}.
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Each set $\Omega_n^i$ is closed under reversal.
Singular words

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**Singular $n$-words of the $i$-th kind**

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That is:

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Each set $\Omega^i_n$ is closed under reversal.

If $w \in \Omega^i_n$ then $w$ is called a singular $n$-word of the $i$-th kind.

Such words play a key role in our study of powers occurring in $s$. 

Powers in a class of $A$-strict standard episturmian words – p.11/17
Powers occurring in $s$

Let $n \in \mathbb{N}^+$ be fixed.

We define $k$ sets of lengths between $|s_n|$ and $|s_{n+1}|$:

$D_1(n) := \{r|s_n| : 1 \leq r \leq d_{n+1}\},$

$D_i(n) := \{|s_n^r s_{n-1}^{d_{n+1-i}} \cdots s_{n+2-i}^{d_{n+3-i}} s_{n+1-i}| : 1 \leq r \leq d_{n+1}\}, \quad 2 \leq i \leq k - 1,$

$D_k(n) := \{|s_n^r s_{n-1}^{d_{n+1-k}} \cdots s_{n+2-k}^{d_{n+3-k}} s_{n+1-k}| : 1 \leq r \leq d_{n+1} - 1\}.$
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$$\mathcal{D}_k(n) := \{|s_n^r s_{n-1}^{d_n} \cdots s_{n+2-k}^{d_{n+3-k}} s_{n+1-k}| : 1 \leq r \leq d_{n+1} - 1\}.$$  

Let $\mathcal{D}_n = \bigcup_{i=1}^k \mathcal{D}_i(n)$.
Let $n \in \mathbb{N}^+$ be fixed.

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$\mathcal{D}_1(n) := \{ r | s_n | : 1 \leq r \leq d_{n+1} \}$,

$\mathcal{D}_i(n) := \{ |s_n s_{n-i}^{d_{n+1} - i} s_{n+1-i}| : 1 \leq r \leq d_{n+1} \}, \quad 2 \leq i \leq k - 1$,

$\mathcal{D}_{k}(n) := \{ |s_n s_{n-1}^{d_{n+1} - k} s_{n+1-k}| : 1 \leq r \leq d_{n+1} - 1 \}$.

Let $\mathcal{D}_n = \bigcup_{i=1}^{k} \mathcal{D}_i(n)$.

Suppose $w \prec s$ and let $p \geq 2$ be an integer. Then,

$$w^p \prec s \Rightarrow |w| \in \mathcal{D}_n \text{ for some } n.$$
Squares

In $s$, successive occurrences of a singular word are positively separated.
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In $s$, successive occurrences of a singular word are positively separated.

Consequently:

**Lemma:** Suppose $u^2 \prec s$ with $|u| \in \mathcal{D}_n$. Then

$$w \not\preceq u \quad \text{if} \quad w \in \Omega_{n+1-i}^1 \text{ for some } i \in [1, k - 1].$$
In $s$, successive occurrences of a singular word are \textit{positively separated}.

Consequently:

\textbf{Lemma:} Suppose $u^2 \prec s$ with $|u| \in D_n$. Then

$$w \not\prec u \quad \text{if} \quad w \in \Omega_{n+1-i}^1 \text{ for some } i \in [1, k - 1].$$

That is, $u$ does not contain a singular $(n + 1 - i)$-word of the first kind for any $i \in [1, k - 1]$. 
Let $w \prec s$ with $|w| \in \mathcal{D}_n$ for some $n$. 
Squares, cubes, and higher powers

Let $w \prec s$ with $|w| \in D_n$ for some $n$.

Our main results show:

If $w^p \prec s$, then $w$ is a conjugate of a finite product of blocks from the set $\{s_n, s_{n-1}, \ldots, s_{n+1-k}\}$, depending on $|w|$ and $d_{n+1}$. 
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For instance:

- Let $p \geq 2$.
- Suppose $|w| = r|s_n|$ for some $r$ with $1 \leq r < (d_{n+1} + 2)/p$. 
Let $w \prec s$ with $|w| \in \mathcal{D}_n$ for some $n$.

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For instance:

- Let $p \geq 2$.
- Suppose $|w| = r|s_n|$ for some $r$ with $1 \leq r < (d_{n+1} + 2)/p$.
- Then:

$$w^p \prec s \iff w \text{ is one of the first } |s_n| \text{ conjugates of } (s_n)^r.$$
Example: $k$-bonacci word

Define the $k$-bonacci word to be the standard episturmian word $\eta_k$ with directive word $(a_1 a_2 \cdots a_k)^\omega$. 
Example: $k$-bonacci word

- Define the $k$-bonacci word to be the standard episturmian word $\eta_k$ with directive word $(a_1a_2\cdots a_k)^\omega$.

- Since all $d_i = 1$, we have $s_n = s_{n-1}s_{n-2}\cdots s_{n-k}$ for all $n \geq 1$.

(The lengths $|s_n|$ are the $k$-bonacci numbers.)
Example: $k$-bonacci word

Define the $k$-bonacci word to be the standard episturmian word $\eta_k$ with directive word $(a_1a_2\cdots a_k)^\omega$.

Since all $d_i = 1$, we have $s_n = s_{n-1}s_{n-2}\cdots s_{n-k}$ for all $n \geq 1$.
(The lengths $|s_n|$ are the $k$-bonacci numbers.)

If $w^p \prec \eta_k$, then

$$|w| = |s_n| + |s_{n-1}| + \cdots + |s_{n+1-i}|$$

for some $n \in \mathbb{N}$ and $i \in [1, k-1]$. 
Our main results reveal that, in $\eta_k$,

- $(a_1)^2$ is the unique square of length 2;
- all conjugates of $s_n$ have a square;
- only some conjugates of $s_n$ have a cube;
- only some conjugates of $s_n s_{n-1} \cdots s_{n+i}$ have a square.
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- $(a_1)^2$ is the unique square of length 2;
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- only some conjugates of $s_n$ have a cube;
- only some conjugates of $s_n s_{n-1} \cdots s_{n+1-i}$ have a square.

There are no other integer powers in $\eta_k$.

In particular, the $k$-bonacci word is 4-power free.
Concluding remarks

Our main results on powers suffice to describe all integer powers occurring in any (episturmian) word that is equivalent to $s$.

Open problem:
Determine all integer powers occurring in general standard episturmian words (with not all $d_i$ necessarily positive).