Fine words over a finite alphabet

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Outline

1. Introduction

2. Preliminaries
   - Finite and Infinite Words
   - Episturmian words

3. Fine Words
   - Definition
   - Previous Results
   - Lemma
   - A Characterization of Fine Words
Let $t$ be an infinite word.

Define $\text{min}(t)$ to be the infinite word such that any prefix of $\text{min}(t)$ is the \textit{lexicographically} smallest amongst the factors of $t$ of the same length.

Similarly define $\text{max}(t)$.

\textbf{Definition (Pirillo 2005)}

An infinite word $t$ over a 2-letter alphabet \{a, b\} ($a < b$) is \textit{fine} if $(\text{min}(t), \text{max}(t)) = (as, bs)$ for some infinite word $s$.

Pirillo (2005) characterized these words.
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Similarly define $\max(t)$.

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An infinite word $t$ over a 2-letter alphabet $\{a, b\}$ ($a < b$) is *fine* if $(\min(t), \max(t)) = (as, bs)$ for some infinite word $s$.

Pirillo (2005) characterized these words.
Here, we:

- **extend** the definition of a fine word to more than two letters;
- **characterize** fine words over a finite alphabet.

### Main Result

An infinite word $t$ is fine $\iff$ $t$ is a *strict episturmian word*, or $t$ is “skew episturmian” (i.e., a particular kind of infinite word, all of whose factors are *episturmian*).
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Let $\mathcal{A}$ be a **finite alphabet** and let $u = x_1x_2 \cdots x_m$, each $x_i \in \mathcal{A}$.

- **Length**: $|u| = m$
- **Reversal**: $\tilde{u} = x_mx_{m-1} \cdots x_1$
- $u$ is a **palindrome** if $u = \tilde{u}$
- $u^\omega$ denotes the **purely periodic** infinite word $uuu \cdots$
- $\mathcal{A}^*$: set of all finite words over $\mathcal{A}$
- $\varepsilon$: the **empty word**
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Words (cont.)

Let \( x \) be an *infinite word* over \( A \).

- **Factor of** \( x \): a finite string of consecutive letters in \( x \)
- **Prefix of** \( x \): factor occurring at the beginning of \( x \)
- \( \Omega(x) \): *set of all factors* of \( x \)
- \( \text{Ult}(x) \): set of letters occurring infinitely often in \( x \)
- \( \text{Alph}(x) \) := \( \Omega(x) \cap A \), the *alphabet* of \( x \)
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Suppose \( \mathcal{A} \) is totally ordered by the relation \(<\). Then we can totally order \( \mathcal{A}^+ \) by the *lexicographic order* \(<\).

That is:

**Definition**

Given two words \( u, v \in \mathcal{A}^+ \), we have \( u < v \iff \) either \( u \) is a proper prefix of \( v \) or \( u = xau' \) and \( v = xbv' \), for some \( x, u', v' \in \mathcal{A}^* \) and letters \( a, b \) with \( a < b \).

- This is the usual alphabetic ordering in a dictionary.
- We say that \( u \) is *lexicographically less* than \( v \).
- This notion naturally extends to infinite words.
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**Sturmian words**

- Infinite words with $n + 1$ distinct factors of length $n$ for each $n \in \mathbb{N}$.
- Sturmian words are over a 2-letter alphabet.
- They are exactly the aperiodic infinite words of minimal complexity.

**Episturmian words**

- A natural generalization of Sturmian words to an arbitrary finite alphabet.
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Definition

An infinite word \( t \) is **episturmian** if:

- \( \Omega(t) \) is \textit{closed under reversal}, and
- \( t \) has at most one \textit{right special factor} of each length.

\( t \) is **standard** if all of its left special factors are prefixes of it.
Standard Episturmian Words

- Let \( t \) be a standard episturmian word over \( A \) and let

\[
U_1 = \varepsilon, \; U_2, \; U_3, \; U_4, \ldots
\]

be the sequence of its palindromic prefixes.

- \( \exists \) an infinite word \( \Delta(t) = x_1x_2x_3\ldots \; (x_i \in A) \) such that

\[
U_{n+1} = (U_nx_n)^{(+), \; n \in \mathbb{N}^+}
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where \( w^{(+)} \) is the shortest palindrome having \( w \) as a prefix.

- \( \Delta(t) \) is called the *directive word* of \( t \).
Let $t$ be a standard episturmian word over $A$ and let

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∃ an infinite word $\Delta(t) = x_1x_2x_3 \cdots (x_i \in A)$ such that

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### Standard Episturmian Words
For each $a \in \mathcal{A}$, define the morphism $\Psi_a$ on $\mathcal{A}$ by

$$\Psi_a : \begin{cases} a & \mapsto a \\ x & \mapsto ax \end{cases} \text{ for all } x \in \mathcal{A} \setminus \{a\}.$$  

All the morphisms $\Psi_a$ generate by composition the monoid of pure epistandard morphisms.

- It includes the identity morphism $\text{Id}_{\mathcal{A}} = \text{Id}$.
- It consists of all the pure standard (Sturmian) morphisms when $|\mathcal{A}| = 2$. 

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A characterization of standard episturmian words

An infinite word $t$ is standard episturmian with $\Delta(t) = x_1x_2x_3 \cdots \ (x_i \in \mathcal{A}) \iff$ there exists an infinite sequence of infinite words $t^{(0)} = t, t^{(1)}, t^{(2)}, \ldots$ such that $t^{(i-1)} = \Psi_{x_i}(t^{(i)})$ for all $i \in \mathbb{N}^+$.

- Each $t^{(i)}$ is a standard episturmian word with $\Delta(t^{(i)}) = x_{i+1}x_{i+2}x_{i+3} \cdots$, the $i$-th shift of $\Delta(t)$.
- Define $\mu_n := \Psi_{x_1}\Psi_{x_2}\cdots\Psi_{x_n}, \quad \mu_0 = \text{Id}$.
- Then, the words

$$h_n := \mu_n(x_{n+1}), \quad n \in \mathbb{N},$$

are clearly prefixes of $t$. 

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**A characterization of standard episturmian words**

**Introduction**

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A characterization of standard episturmian words

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Strict episturmian words

**Definition**

A standard episturmian word $t$ over $A$, or any equivalent (episturmian) word, is said to be $B$-strict (or $k$-strict if $|B| = k$) if

$$\text{Alph}(\Delta(t)) = \text{Ult}(\Delta(t)) = B \subseteq A$$

- The $k$-strict episturmian words have complexity $(k - 1)n + 1$ for each $n \in \mathbb{N}$.
- Such words are exactly the $k$-letter Arnoux-Rauzy sequences.
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An infinite word $t$ is **fine** if there exists an infinite word $s$ such that, for any letter $a \in \text{Alph}(t)$ and order $<$ such that $a = \min(\text{Alph}(t))$, we have $\min(t) = as$.

Proposition (Pirillo 2005)

Let $t$ be an infinite word over $\{a, b\}$. The following properties are equivalent:

(i) $t$ is fine,

(ii) either $t$ is a Sturmian word, or $t = v\mu(x)^\omega$ where $\mu$ is a pure standard Sturmian morphism on $\{a, b\}$, and $v$ is a non-empty suffix of $\mu(x^p y)$ for some $p \in \mathbb{N}$ and $x, y \in \{a, b\}$ ($x \neq y$).
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Proposition (Justin & Pirillo 2002)

Let $s$ be an infinite word over a finite alphabet $\mathcal{A}$. The following properties are equivalent:

(i) $s$ is a standard Arnoux-Rauzy sequence,

(ii) for any $a \in \mathcal{A}$ and order $<$ such that $a = \min(\mathcal{A})$, we have $as = \min(s)$.

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Outline

1. Introduction

2. Preliminaries
   - Finite and Infinite Words
   - Episturmian words

3. Fine Words
   - Definition
   - Previous Results
   - Lemma
   - A Characterization of Fine Words

Fine words
Lemma

Let \( \mathcal{A} \) be a finite alphabet and let \( a \in \mathcal{A} \).
Suppose \( t, s \) are infinite words over \( \mathcal{A} \) such that

\[
t = \Psi_z(t^{(1)}) \text{ and } s = \Psi_z(s^{(1)}) \text{ for some } z \in \text{Alph}(t^{(1)}).\]

Then \( \min(t^{(1)}) = as^{(1)} \iff \min(t) = as \).

Remark

- If \( z \) is any letter (not necessarily in \( \text{Alph}(t^{(1)}) \)), then

\[
\min(t^{(1)}) = as^{(1)} \iff \min(t) = \begin{cases} zas & \text{if } z < a, \\ as & \text{if } z \geq a. \end{cases}
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Let $\mathcal{A} = \{a, b, c\}$ with $a < b < c$.

**Example (1)**

Suppose $f$ is the Fibonacci word over $\{a, b\}$ (i.e., the standard episturmian word directed by $(ab)^\omega$).

- Then $\min(f) = af$.
- Hence $\min(\psi_c(f)) = a\psi_c(f)$.

**Example (2)**

Suppose $f'$ is the Fibonacci word over $\{b, c\}$.

- Then $\min(f') = bf'$.
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Notation
Let \( x_p \) denote the prefix of length \( p \) of an infinite word \( x \).

Theorem
Let \( t \) be an infinite word with \( \text{Alph}(t) = A \).
Then, \( t \) is fine if and only if one of the following holds:

(i) \( t \) is a strict episturmian word;
(ii) \( t = v\mu(f) \) where \( f \) is a \( B \)-strict standard episturmian word with \( B = A \setminus \{x\} \), \( \mu \) is a pure epistandard morphism on \( A \), and \( v \) is a non-empty suffix of \( \mu(\tilde{f}_p x) \) for some \( p \in \mathbb{N} \).
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