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ROTUNDITY REDUX

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Abstract

Recently the concept of uniform rotundity was generalized for real Banach spaces by using a type of "area" devised for these spaces. This paper modifies the methods used for uniform rotundity and applies them to weak rotundity in real and complex spaces. This leads to the definition of k-smoothness, k-very smoothness and k-strong smoothness. As an application, several sufficient conditions for reflexivity are obtained.

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1. Introduction

Sullivan (1979) has generalized uniform rotundity by using the idea of "area" given in Silverman (1951) and Geremia and Sullivan (1981). In this paper I will also use this idea of "area" when generalizing weak rotundity for real and complex Banach spaces. These same methods are then used to generalize ordinary rotundity and smoothness. In order to do this the structure of higher duals must be investigated: this leads to an extension of a theorem of Dixmier (1948; page 1070) concerning the shape of the unit sphere in these duals.

One of the reasons for studying smoothness in Banach spaces is that it can often give some information about reflexivity. It is well known that if a real or complex Banach space has a very smooth first dual, or if a real space has a dual space with a Fréchet differentiable norm, then the space must be reflexive. It is shown here that the generalization of these concepts, k-very smoothness and

k-strong smoothness, also give sufficient conditions for reflexivity: k-very smoothness in the appropriate higher dual (Theorem 4.3) and k-strong smoothness in the first dual for all $k \geqslant 1$ (Theorem 5.1).

2. Preliminaries

Let E be a real or complex Banach space and $E^{(n)}$ its nth dual, $n \geqslant 0$ ($E^{(0)} = E$). $E^{(1)}$ and $E^{(2)}$ will usually be denoted by E^* and E^{**}, respectively. The unit sphere of $E^{(n)}$, the set $\{x \in E^{(n)}: \|x\| = 1\}$, will be denoted by $S(E^{(n)})$. The set-valued mapping D_n: $S(E^{(n)}) \to S(E^{(n+1)})$ which associates each $x \in S(E^{(n)})$ with the set $\{f \in S(E^{(n+1)}): f(x) = 1\}$ is called the *duality mapping*. $D_n(S(E^{(n)}))$ will be denoted by $D_n(S)$, $[D_0(S) = D(S)]$.

Let Q_n denote the natural embedding of $E^{(n)}$ into $E^{(n+2)}$, $n \geqslant 0$. $Q_n E^{(n)}$ and $Q_n(Q_{n-2}E^{(n-2)})$ will be denoted by $\hat{E}^{(n)}$ and $\hat{\hat{E}}^{(n-2)}$, respectively. The linear isometry Q_n^{**} maps $E^{(n+2)}$ into $E^{(n+4)}$ with $Q_n^{**}(E^{(n+2)}) \cap \hat{E}^{(n+2)} = \hat{\hat{E}}^{(n)}$.

All definitions and proofs in this paper are expressed in terms of sequences. Generally, the modifications needed in order to replace "sequence" by "net" are minimal.

I would like to thank the referee for his suggestions, especially those which led to Example 4.2.

3. k-weak rotundity and k-smoothness

Let f, $g \in S(E^*)$. E is *weakly rotund at, or with respect to, f in the g direction* if and only if for any pair of sequences $\{x_n\}$, $\{y_n\}$ of elements of $S(E)$ with $f(x_n + y_n) \to 2$, we have $g(x_n - y_n) \to 0$. [See Cudia (1964) and Yorke (1977).] Geometrically, this means that if the lengths of the lines in $\mathbf{R}$ (or $\mathbf{C}$) joining the points $g(x_n)$ and $g(y_n)$ remain bounded away from zero, then the sequence $\{(x_n + y_n)/2\}$ cannot approach the hyperplane $f^{-1}(1)$. As in Sullivan (1979) this can be generalized as follows: Let $f, g_1, g_2, \ldots, g_k \in S(E^*)$. E will be said to be *k-weakly rotund with respect to f in the $g_1, \ldots, g_k$ directions* if and only if for any $k+1$ sequences $\{x_n^1\}, \{x_n^2\}, \ldots, \{x_n^{k+1}\}$ of elements of $S(E)$ with

$$f\left(x_n^1 + x_n^2 + \cdots + x_n^{k+1}\right) \to k+1,$$

we have

$$A\left(x_n^1, \ldots, x_n^{k+1}; g_1, \ldots, g_k\right) = \operatorname{abs}\left(\frac{1}{k!} \begin{vmatrix} 1 & 1 & & 1 \\ g_1(x_n^1) & g_1(x_n^2) & \cdots & g_1(x_n^{k+1}) \\ \vdots & & & \vdots \\ g_k(x_n^1) & g_k(x_n^2) & \vdots & g_k(x_n^{k+1}) \end{vmatrix} \right) \to 0,$$

as $n \to \infty$. Here $|\cdot|$ is the usual determinant function and "abs" denotes the absolute value function. In other words, E is k-weakly rotund with respect to f in the $g_1, g_2, \ldots, g_k$ directions if and only if whenever the k-dimensional "area" of each of the figures in $\mathbf{R}^k$ (or $\mathbf{C}^k$) enclosed by the $k+1$ points $P_n^1 = (g_1(x_n^1), g_2(x_n^1), \ldots, g_k(x_n^1))$, $P_n^2 = (g_1(x_n^2), \ldots, g_k(x_n^2)), \ldots$, $P_n^{k+1} = (g_1(x_n^{k+1}), \ldots, g_k(x_n^{k+1}))$ is bounded away from zero, then the sequence $\{(x_n^1 + x_n^2 + \cdots + x_n^{k+1})/(k+1)\}$ of elements of the unit ball of E does not approach the hyperplane $f^{-1}(1)$. Clearly, E is k-weakly rotund with respect to f in the $g_1, g_2, \ldots, g_k$ directions if and only if E is k-weakly rotund with respect to f in the $h_1, h_2, \ldots, h_k$ directions, $h_i \in S(E^*)$ and $1 \leqslant i \leqslant k$, for any set $\{h_1, h_2, \ldots, h_k\} \subset \overline{\mathrm{sp}}\{g_1, g_2, \ldots, g_k\}$. Thus, in a sense, k-weak rotundity with respect to a particular $f \in S(E^*)$ describes a geometrical property of E with respect to the set of k-dimensional subspaces of E^*.

Now let A and B be non-empty subsets of $S(E^*)$. E is *k-weakly rotund with respect to A in the B directions* if and only if E is k-weakly rotund with respect to f in the $g_1, g_2, \ldots, g_k$ directions for each $f \in A$ and each subset of k elements $\{g_1, g_2, \ldots, g_k\} \subset B$. When E^* is being considered the sets A and B will be subsets of either $S(E^{**})$ or $S(\hat{E})$.

If E is k-weakly rotund uniformly with respect to $S(E^*)$ uniformly in the $S(E^*)$ directions, then E is *k-uniformly rotund* (k-UR) [see Yorke (1977; pages 225–226)]. When E is k-weakly rotund with respect to $f \in S(E^*)$ uniformly in the $S(E^*)$ directions, E will be said to be k-UR *with respect to f*. This property will be investigated more fully in Section 5.

A Banach space E, either real or complex, is *smooth at $x \in S(E)$* if and only if whenever f_1 and f_2 are in $D(x)$, the set $\{f_1, f_2\}$ is linearly dependent. Since $f_1(x) = f_2(x) = 1$, the set $\{f_1, f_2\}$ is linearly dependent if and only if $f_1 = f_2$; thus this definition of smoothness is equivalent to the usual one.

E will be said to be *k-smooth at $x \in S(E)$* if and only if whenever $f_1, f_2, \ldots, f_{k+1}$ are in $D(x)$, the set $\{f_1, f_2, \ldots, f_{k+1}\}$ is linearly dependent. E is *k-smooth* if E is k-smooth at each $x \in S(E)$. If E^{**} is k-smooth at $\hat{x} \in S(\hat{E})$, then E will be said to be *k-very smooth at $x \in S(E)$*. E is *k-very smooth* if E is k-very smooth at each $x \in S(E)$.

Let $x_1, x_2, \ldots, x_{k+1}$ be arbitrary points of $S(E)$. The k-dimensional "area" of the figure enclosed by these $k+1$ points in $S(E)$, denoted by $A(x_1, x_2, \ldots, x_{k+1})$, is defined as

$$\sup\left\{\mathrm{abs}\left(\frac{1}{k!}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ f_1(x_1) & f_1(x_2) & & f_1(x_{k+1}) \\ \vdots & & & f_k(x_{k+1}) \\ f_k(x_1) & f_k(x_2) & \vdots & \end{vmatrix}\right) : f_i \in S(E^*), 1 \leqslant i \leqslant k\right\}.$$

[See Silverman (1951) and Sullivan (1979).] The following lemma appears in Geremia and Sullivan (1981; page 233).

LEMMA 3.1. *Let* $f_1, f_2, \dots, f_{k+1} \in D(x)$, $x \in S(E)$. *The set* $\{f_1, f_2, \dots, f_{k+1}\}$ *is linearly independent if and only if* $A(f_1, f_2, \dots, f_{k+1}) > 0$.

PROOF. Since $A(f_1, f_2, \dots, f_{k+1}) \geqslant \operatorname{dist}(f_{k+1}, \overline{\operatorname{sp}}\{f_1, f_2, \dots, f_k\}) \cdot A(f_1, f_2, \dots, f_k)$, this means $A(f_1, f_2, \dots, f_{k+1}) > 0$ if $\{f_1, f_2, \dots, f_{k+1}\}$ is linearly independent. If $\{f_1, f_2, \dots, f_{k+1}\}$ is linearly dependent, then $f_{k+1} = \sum_{j=1}^{k} \alpha_j f_j$ with the scalars α_j not all zero. However, $f_j(x) = 1 = f_{k+1}(x)$, $1 \leqslant j \leqslant k$, so $\sum_{j=1}^{k} \alpha_j = 1$. Now standard manipulation of the determinant function gives $A(f_1, f_2, \dots, f_{k+1}) = 0$.

THEOREM 3.1. 1. *E is k-smooth at $x \in S(E)$ if and only if E^* is k-weakly rotund with respect to $\hat{x}$ in the $S(\hat{E})$ directions.*

2. *E is k-very smooth at $x \in S(E)$ if and only if E^* is k-weakly rotund with respect to $\hat{x}$ in the $S(E^{**})$ directions.*

PROOF. 1. If E is not k-smooth at $x \in S(E)$, then there are $f_1, f_2, \dots, f_{k+1} \in D(x)$ such that the set $\{f_1, f_2, \dots, f_{k+1}\}$ is linearly independent. Hence, by Lemma 3.1, $A(f_1, f_2, \dots, f_{k+1}) > 0$. This means there must be a set $\{y_1, y_2, \dots, y_k\} \subset S(E)$ such that $A(f_1, f_2, \dots, f_{k+1}; \hat{y}_1, \hat{y}_2, \dots, \hat{y}_k) > 0$. Therefore since $(f_1 + f_2 + \cdots + f_{k+1})(x) = k + 1$, E^* cannot be k-weakly rotund with respect to $\hat{x}$ in the $\hat{y}_1, \hat{y}_2, \dots, \hat{y}_k$ directions.

Conversely, assume that E^* is not k-weakly rotund with respect to $\hat{x}$ in the $S(\hat{E})$ directions. Then there is a set

$$\{\hat{y}_1, \hat{y}_2, \dots, \hat{y}_k\} \subset S(\hat{E})$$

and sequences $\{f_n^1\}, \{f_n^2\}, \dots, \{f_n^{k+1}\}$ of elements of $S(E^*)$ such that $A(f_n^1, f_n^2, \dots, f_n^{k+1}; \hat{y}_1, \hat{y}_2, \dots, \hat{y}_k)$ remains bounded away from zero for all n even though $(f_n^1 + f_n^2 + \cdots + f_n^{k+1})(x) \to k + 1$ as $n \to \infty$. Let $g_1, g_2, \dots, g_{k+1}$ be $\sigma(E^*, E)$ cluster points of $\{f_n^1\}, \{f_n^2\}, \dots, \{f_n^{k+1}\}$, respectively. Since $f_n^i(x) \to 1$, each $g_i \in D(x)$, $1 \leqslant i \leqslant k + 1$; thus $A(g_1, g_2, \dots, g_{k+1}) > 0$, so, by Lemma 3.1, E is not k-smooth at x.

2. This is proved similarly.

THEOREM 3.2. *The following are equivalent*:

1. *E^* is k-smooth at $f \in S(E^*)$;*
2. *E is k-weakly rotund with respect to f in the $S(E^*)$ directions;*
3. *E^{**} is k-weakly rotund with respect to f in the $S(E^*)$ directions.*

PROOF. By Theorem 3.1.1 it is sufficient to show that (2) implies (3). Assume that (3) does not hold. Then there is an $\varepsilon > 0$, a set $\{g_1, g_2, \ldots, g_k\} \subset S(E^*)$, and $k + 1$ sequences $\{F_n^1\}, \{F_n^2\}, \ldots, \{F_n^{k+1}\}$ of elements of $S(E^{**})$ such that

$$\left(F_n^1 + F_n^2 + \cdots + F_n^{k+1}\right)(f) \to k + 1 \qquad \text{as } n \to \infty$$

and $A(F_n^1, F_n^2, \ldots, F_n^{k+1}; g_1, g_2, \ldots, g_k) \geqslant \varepsilon$ for all n. For each j, $1 \leqslant j \leqslant k + 1$, and each n let V_n^j be the $\sigma(E^{**}, E^*)$ neighbourhood of F_n^j determined by $f, g_1, g_2, \ldots, g_k$ and $1/n$. Now use the "weak-*" density of $S(\hat{E})$ in $S(E^{**})$ to construct $k + 1$ sequences $\{x_n^1\}, \{x_n^2\}, \ldots, \{x_n^{k+1}\}$ of elements of $S(E)$ with the following properties: $f(x_n^1 + x_n^2 + \cdots + x_n^{k+1}) \to k + 1$ as $n \to \infty$ and $A(x_n^1, x_n^2, \ldots, x_n^{k+1}; g_1, g_2, \ldots, g_k)$ remains bounded away from zero for all n. Thus (2) fails to hold.

It is clear from the definitions that if E is k-smooth, then E is $(k + 1)$-smooth. Thus, by Theorem 3.2, if E is k-weakly rotund, then E is $(k + 1)$-weakly rotund as well.

4. k-rotundity and reflexivity

E is *rotund*, or *1-rotund*, if and only if whenever $x, y \in S(E)$ and $\|x + y\| = 2$, then the set $\{x, y\}$ is linearly dependent. As with smoothness, it is easy to show that this definition is equivalent to the usual one. E will be said to be *k-rotund* if and only if whenever $x_1, x_2, \ldots, x_{k+1} \in S(E)$ and $\|x_1 + x_2 + \cdots + x_{k+1}\| = k + 1$, then the set $\{x_1, x_2, \ldots, x_{k+1}\}$ is linearly dependent. Certainly, if E is k-rotund, then E must be $(k + 1)$-rotund.

THEOREM 4.1. 1. *If E^* is k-smooth, then E is k-rotund.*
2. *If E^* is k-rotund, then E is k-smooth.*

This follows directly from the definitions.

EXAMPLE 4.1. Let E be l_1 with its usual norm and e_i denote the vector $(0, \ldots, 0, 1, 0 \ldots)$ with "1" in the ith position and zero elsewhere. For any fixed k, the set $\{e_i\colon 1 \leqslant i \leqslant k + 1\}$ of unit vectors is linearly independent and $\|e_1 + e_2 + \cdots + e_{k+1}\| = k + 1$. Thus l_1 is not k-rotund for any k, so, by Theorem 4.1, l_∞ is not k-smooth for any k.

Now before proceeding any further, some properties of higher duals must be investigated. First, choose an F from $S(E^{**}) \setminus S(\hat{E})$ and consider the following sequence [see Perrott (1979)]: $F_1 = F$, $F_2 = Q_0^{**}F$, $F_3 = Q_2^{**}F_2 (= Q_2^{**}Q_0^{**}F), \ldots, F_{k+1} = Q_{2k-2}^{**}F_k$. For each j, $1 \leqslant j \leqslant k$, $F_j \in S(E^{(2j)})$, while

$F_{j+1} \in S(E^{(2j+2)}) \setminus \hat{S}(E^{(2j)})$. (This will be used in the proof of Lemma 4.1.) Therefore the set $\{F_1, F_2, \ldots, F_{k+1}\}$, which can be considered to be embedded in $E^{(2k+2)}$ under the natural embeddings, is a linearly independent set of vectors in $S(E^{(2k+2)})$ for each $k \geqslant 1$.

Let $\{f_n\}$ be a sequence in $S(E^*)$ such that $F(f_n) \to \|F\| = 1$ as $n \to \infty$. Using the properties of the operators Q_{2k-2}^{**}, $k \geqslant 1$, it is easy to show that $F_j(f_n) \to \|F_j\| = 1$ for each j, $1 \leqslant j \leqslant k+1$, and $\|F_1 + F_2 + \cdots + F_{k+1}\| = k+1$. This leads to the following generalization of a theorem by Dixmier (1948; Théorème 20) which appears in Geremia and Sullivan (1981).

THEOREM 4.2. *If E is a non-reflexive space, then $E^{(2k+2)}$ and $E^{(2k+3)}$ fail to be k-rotund for each $k \geqslant 1$.*

NOTATION. Let $\widetilde{E^*}$ denote the natural embedding of E^* in $E^{(2k-1)}$ and $\widetilde{E^{**}}$ denote the natural embedding of E^{**} in $E^{(2k)}$, $k > 1$.

THEOREM 4.3. *If $\widetilde{E^*}$ or $\widetilde{E^{**}}$ is k-very smooth, then E is reflexive.*

The proof of this result rests on the following lemma:

LEMMA 4.1. *Let $F \in S(E^{**})$ and $F_1, F_2, \ldots, F_{k+1}$ be as above. If $\{F_1, F_2, \ldots, F_{k+1}\}$ is a linearly dependent set for some $k \geqslant 1$, then $F = \hat{x}$ for some $x \in S(E)$.*

PROOF. If $\{F_1, F_2, \ldots, F_{k+1}\}$ is a linearly dependent set, then $F_{k+1} \in S(\hat{E}^{(2k)}) \subset S(E^{(2k+2)})$. But since $F_{k+1} = Q_{2k-2}^{**} F_k$ and $Q_{2k-2}^{**}(E^{(2k)}) \cap \hat{\hat{E}}^{(2k)}$, $F_{k+1} \in S(E^{(2k-2)})$. Thus since $F_k \in S(E^{(2k-2)})$, $F_{k+1} = F_k$. Continuing in this way gives $F_2 = F$, with $F \in \hat{\hat{S}}(E)$; that is, $F = \hat{x}$ for some $s \in S(E)$.

PROOF OF THE THEOREM. If E is assumed to be non-reflexive, then it is possible to choose an f from $S(E^*) \setminus D(S)$. Let $F \in D_1(f)$; obviously $F \notin S(\hat{E})$. Since $\widetilde{E^*}$ is k-very smooth, and $F_1 = F$, $F_2 = Q_0^{**} F_1, \ldots, F_{k+1} = Q_{2k-2}^{**} F_k$ are all in $D_{2k+1}(\tilde{f}) \subset S(E^{(2k+2)})$, the set $\{F_1, F_2, \ldots, F_k\}$ must be linearly dependent. But, by Lemma 4.1, this means that $F = \hat{x}$, therefore f must be in $D(S)$. Since f was chosen arbitrarily, $D(S) = S(E^*)$, hence E is reflexive. The proof for the case when $\widetilde{E^{**}}$ is k-very smooth is identical.

EXAMPLE 4.2. A dual space E^* is said to be *weak-* uniformly rotund* (W^*UR) if it is weakly rotund uniformly with respect to $S(E)$ in the $S(E)$ directions; that is, if for any pair of sequences $\{f_n\}$, $\{g_n\}$ of elements of $S(E^*)$ with $\|f_n + g_n\| \to 2$, we have $(f_n - g_n)(x) \to 0$ for each $x \in S(E)$ as $n \to \infty$. Let J denote the *James space* (James; 1951). J is *quasi-reflexive* of order one; that is, $\dim(J^{**}/\hat{J}) = 1$ [Civin and Yood (1957)]. J^* is separable, hence J can be equivalently renormed so that J^{**} is W^*UR [Zizler (1968; page 429)]. Since J is not reflexive, $J^{(4)}$ cannot be rotund and $J^{(3)}$ cannot be smooth. The aim now is to show that $J^{(4)}$, so renormed, is 2-rotund.

Let $X_i^{(4)} = \hat{F}_i + \alpha_i x^{*\perp}$, $i = 1, 2, 3$, α_i scalars, be any three elements of $S(J^{(4)})$ with $\|X_1^{(4)} + X_2^{(4)}\| = \|X_2^{(4)} + X_3^{(4)}\| = \|X_1^{(4)} + X_3^{(4)}\| = 2$. (Here $J^{(4)} = \hat{J}^{**} + \mathrm{sp}\{x^{*\perp}\}$.) Choose a sequence $\{\mathscr{F}_n\}$ of elements of $S(J^{(3)})$ such that $(X_1^{(4)} + X_2^{(4)})(\mathscr{F}_n) \to \|X_1^{(4)} + X_2^{(4)}\| = 2$, and an arbitrary element f from $S(J^*)$. For each n construct the $\sigma(J^{(4)}, J^{(3)})$ neighborhoods of $X_1^{(4)}$ and $X_2^{(4)}$ determined by $\mathscr{F}_n$, $\hat{f}$, and $1/n$. The usual "weak-* density" argument now gives sequences $\{G_n\}$ and $\{H_n\}$ of elements of $S(J^{**})$ such that $\|G_n + H_n\| \to 2$, $(X_1^{(4)} - \hat{G}_n)(\hat{f}) \to 0$, and $(X_2^{(4)} - \hat{H}_n)(\hat{f}) \to 0$ as $n \to \infty$. But J^{**} is W^*UR, so $(G_n - H_n)(f) \to 0$; hence $X_1^{(4)}(\hat{f}) = X_2^{(4)}(\hat{f})$. However, f was chosen arbitrarily so $X_1^{(4)} = X_2^{(4)}$ for all $f \in S(J^*)$; that is, $F_1 = F_2$. The same procedure applied to $X_2^{(4)}$ and $X_3^{(4)}$ gives $F_2 = F_3$. Thus the vectors $X_1^{(4)}$, $X_2^{(4)}$, and $X_3^{(4)}$ are collinear, so $J^{(4)}$ must be 2-rotund. Now applying Theorem 4.1 (1) gives that $J^{(3)}$ is 2-smooth.

5. k-strong smoothness

Let E be a real or complex space. E will be said to be *k-strongly smooth at* $x \in S(E)$ if and only if E^* is k-UR with respect to $\hat{x}$; that is, if $\{f_n^1\}$, $\{f_n^2\}, \ldots, \{f_n^{k+1}\}$ are $k+1$ sequences of elements of $S(E^*)$ and $(f_n^1 + f_n^2 + \cdots + f_n^{k+1})(x) \to k+1$, then $A(f_n^1, f_n^2, \ldots, f_n^{k+1}) \to 0$ as $n \to \infty$. Geometrically this means that E^* is k-UR with respect to $\hat{x}$ (E is k-strongly smooth at x) if and only if whenever the k-dimensional "areas" of the figures in the unit ball of E^* enclosed by $f_n^1, f_n^2, \ldots, f_n^{k+1}$ remain bounded away from zero, then the sequence $(f_n^1 + f_n^2 + \cdots + f_n^{k+1})/k + 1$ of centroids of these figures does not approach the hyperplane $\hat{x}^{-1}(1)$ as $n \to \infty$. E is *k-strongly smooth* if E is k-strongly smooth at x for each $x \in S(E)$. When $k = 1$, E will be said to be strongly smooth rather than 1-strongly smooth.

If E is a real space, then E is strongly smooth if and only if the norm of E is Fréchet differentiable at each $x \in S(E)$ [Yorke (1977; Proposition 5)]. Notice, however, that k-strong smoothness is defined for both real and complex spaces as well as for $k \geqslant 1$.

LEMMA 5.1. *E is k-strongly smooth at $x \in S(E)$ if and only if E^{**} is k-strongly smooth at $\hat{x}$.*

PROOF. It is sufficient to show that if E^* is k-UR with respect to $\hat{x}$, then $E^{(3)}$ is k-UR with respect to $\hat{x}$ (see Theorem 3.2). Assume otherwise. Then there is an $\varepsilon > 0$, and $k + 1$ sequences $\{\mathscr{F}_n^1\}, \{\mathscr{F}_n^2\}, \ldots, \{\mathscr{F}_n^{k+1}\}$ of elements of $S(E^{(3)})$ such that even though $(\mathscr{F}_n^1 + \mathscr{F}_n^2 + \cdots + \mathscr{F}_n^{k+1})(\hat{x}) \to k + 1$ as $n \to \infty$, $A(\mathscr{F}_n^1, \mathscr{F}_n^2, \ldots, \mathscr{F}_n^{k+1}) \geqslant \varepsilon$ for all n. This means that for each (fixed) n there are sequences $\{F_{m,n}^1\}, \{F_{m,n}^2\}, \ldots, \{F_{m,n}^{k+1}\}$ of elements of $S(E^{**})$ with the property that $A(\mathscr{F}_n^1, \ldots, \mathscr{F}_n^{k+1}; F_{m,n}^1, \ldots, F_{m,n}^{k+1})$ remains bounded away from zero for all m. For each j, $1 \leqslant j \leqslant k + 1$, let $\{F_n^j\}$ denote the diagonal sequence $(F_{n,n}^n)$ and V_n^j the $\sigma(E^{(3)}, E^{(2)})$ neighbourhood of $\mathscr{F}_n^j$ determined by $\hat{x}, F_n^1, F_n^2, \ldots, F_n^{k+1}$ and $1/n$. The standard "weak-* density" argument now gives $k + 1$ sequences $\{f_n^1\}$, $\{f_n^2\}, \ldots, \{f_n^{k+1}\}$ of elements of $S(E^*)$ with $\hat{f}_n^j \in V_n^j$ for each n and each j. Thus $(f_n^1 + f_n^2 + \cdots + f_n^{k+1})(x) \to k + 1$ as $n \to \infty$, but $A(f_n^1, f_n^2, \ldots, f_n^{k+1}) \geqslant A(f_n^1, f_n^2, \ldots, f_n^{k+1}; F_n^1, F_n^2, \ldots, F_n^{k+1}) > 0$ for all n. Therefore E^* is not k-UR with respect to $\hat{x}$.

THEOREM 5.1. *If E^* or E^{**} is k-strongly smooth for any $k \geqslant 1$, then E is reflexive.*

PROOF. If E^* (E^{**}) is k-strongly smooth for some k, then, by Lemma 5.1, so is $\widetilde{E^*}$ ($\widetilde{E^{**}}$). The result now folllows from Theorem 4.3.

Consequently, if E is a real space, then E^* (E^{**}) is k-strongly smooth for some $k \geqslant 1$ if and only if E is isomorphic to a space whose dual has a Fréchet differentiable norm.

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