SETS OF $p$-SPECTRAL SYNTHESIS

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Let $G$ be a Hausdorff locally compact Abelian group, $\Gamma$ its character group. Certain closed subsets of $\Gamma$ are introduced, these being closely related to sets of spectral synthesis for $L^1(G)$. Some properties and examples of these sets are discussed, and then a Malliavin-type result is obtained.

In general we follow the notation used in [1]. We shall let $\lambda, \theta$ denote Haar measures on $G, \Gamma$ respectively, chosen so that Plancherel's theorem holds.

1. The definition and some properties of $S_p$- and $C_p$-sets.

**Definition 1.1.** Let $\Xi$ be a closed subset of $\Gamma$. We shall call $\Xi$ an $S_p$-set ($p \in [1, \infty)$) if, given $\epsilon > 0$ and $f \in L^1 \cap L^p(G)$ such that $\hat{f}$ vanishes on $\Xi$, there exists $g \in L^1 \cap L^p(G)$ such that $\hat{g}$ vanishes on a neighbourhood of $\Xi$ and $\|f - g\|_p < \epsilon$. If such a $g$ can be found of the form $h \ast f$, where $h \in L^1(G)$ and $\hat{h}$ vanishes on a neighbourhood of $\Xi$, then $\Xi$ will be called a $C_p$-set. We also define $S_\infty$- and $C_\infty$-sets as above, with $f, g$ in $V \cap C_0(G)$ (rather than $V \cap L^\infty(G)$).

Since, by [1], (33.12), $L^1(G)$ admits a bounded positive approximate identity $\{u_i\}_{i \in I}$ such that for each $i \in I$, $u_i \in L^1 \cap C_0(G)$ and supp($\hat{u}_i$) is compact, it follows (see [1], (32.33) (b) and (32.48) (a)) that we can (and shall) assume in Definition 1.1 that $f, g, h \in L^1 \cap C_0(G)$, where supp($\hat{f}$) is compact and both supp($\hat{g}$) and supp($\hat{h}$) are compact and disjoint from $\Xi$ ($p \in [1, \infty)$).

Clearly every $C_p$-set is an $S_p$-set. For the case $p = 1$ we just have the familiar $S$-set and $C$-set; see [3], 7.2.5 (a) and 7.5.1 respectively.

For $f \in L^\infty(G)$ the spectrum (written $\Sigma(f)$) will be defined as in [1], (40.21). For $f \in L^p(G)$ ($p \in [1, \infty)$), we define its spectrum by

$$\Sigma(f) = \bigcup \{\Sigma(\phi \ast f) : \phi \in C_0(G)\}$$

It is easily proved that for $f \in L^1(G)$, $\Sigma(f) = \text{supp}(\hat{f})$.

Given $\Xi \subseteq \Gamma$, we write

$$L^p_\Xi(G) = \{f \in L^p(G) : \Sigma(f) \subseteq \Xi\}.$$
We now have the following characterisation of $S_p$- and $C_p$-sets:

**Theorem 1.2.** Let $p \in [1, \infty)$ and suppose $\Xi$ is a closed subset of $\Gamma$. Then

(a) $\Xi$ is an $S_p$-set if and only if for all $l \in L^p_\infty(G)$ and for all $f \in L^1 \cap C_0(G)$ such that $\text{supp}(\hat{f})$ is compact and $\hat{f}$ vanishes on $\Xi$, we have $l \ast f = 0$;

(b) $\Xi$ is a $C_p$-set if and only if for all $f \in L^1 \cap C_0(G)$ such that $\text{supp}(\hat{f})$ is compact and $\hat{f}$ vanishes on $\Xi$, and for all $l \in L^p_\infty(G)$ such that $l \ast f \in L^p_\infty(G)$, we have $l \ast f = 0$.

This result is known for the case $p = 1$ (see [2], Chapter 7, 1.2 and 4.9). The proof is standard, and we shall not include it.

It is easy to adapt the proof of [3], Theorem 7.5.2 to give:

**Theorem 1.3.** Let $p \in [1, \infty]$. Then

(a) every one-point subset of $\Gamma$ is a $C_p$-set in $\Gamma$;

(b) finite unions of $C_p$-sets in $\Gamma$ are $C_p$-sets in $\Gamma$;

(c) if the boundary of a closed set $\Xi$ is a $C_p$-set, so is $\Xi$;

(d) if $\Xi$ is a closed subset of a closed subgroup $\Lambda$ of $\Gamma$, if $\partial_\Lambda(\Xi)$ is the boundary of $\Xi$ relative to $\Lambda$, and if $\partial_\Lambda(\Xi)$ is a $C_p$-set in $\Gamma$ then $\Xi$ is also a $C_p$-set in $\Gamma$;

(e) each closed subgroup of $\Gamma$ is a $C_p$-set in $\Gamma$.

For $p \in [1, 2]$ it is not known whether the notions of $C_p$-set and $S_p$-set are identical (it appears in Theorem 2.1 that every closed set is a $C_p$-set for $p \geq 2$). Furthermore we cannot say whether the union of two $S_p$-sets is itself an $S_p$-set. We can however obtain two partial results in this direction. Both these results (Theorem 1.4 (a), (b)) are known for the case $p = 1$ (see [2], Chapter 2, 7.5).

**Theorem 1.4.** (a) Suppose $\Xi = \Xi_1 \cup \Xi_2$, where $\Xi_1$ and $\Xi_2$ are disjoint closed subsets of $\Gamma$. Then, for $p \in [1, \infty)$, $\Xi$ is an $S_p$-set if and only if both $\Xi_1$ and $\Xi_2$ are $S_p$-sets.

(b) Let $p \in [1, \infty)$ and suppose $\Xi_1$ is an $S_p$-set and $\Xi_2$ is a $C_p$-set. Then $\Xi = \Xi_1 \cup \Xi_2$ is an $S_p$-set.

The final result of this section gives us an inclusion result between the set of $C_p$-sets (respectively $S_p$-sets) and the set of $C_q$-sets (respectively $S_q$-sets) for $1 \leq p < q \leq \infty$.

**Theorem 1.5.** Let $1 \leq p < q \leq \infty$. Then every $C_p$-set (respectively $S_p$-set) is a $C_q$-set (respectively $S_q$-set).
Proof. Assume $\Xi$ is a $C_p$-set. Suppose we are given $\epsilon > 0$ and $f \in L^1 \cap C_0(G)$ with $\text{supp}(\hat{f})$ compact and $\hat{f}$ vanishing on $\Xi$. We can find $h \in L^1 \cap C_0(G)$ such that $\|f - h \ast f\|_q < \epsilon/2$. Since $\Xi$ is a $C_p$-set there exists $g \in L^1(G)$ such that $\hat{g}$ has compact support disjoint from $\Xi$ and $\|h\|_r \|\hat{g} \ast f\|_p < \epsilon/2$, where $p^{-1} + r^{-1} - q^{-1} = 1$ (with the usual convention for the cases $p = 1$ and $q = \infty$). Now (see [1], (20.18))

$$\|f - h \ast g \ast f\|_q \leq \|f - h \ast f\|_q + \|h\|_r \|\hat{g} \ast f\|_p + \epsilon.$$ 

It remains only to note that $h \ast g \in L^1 \cap C_0(G)$ and $(h \ast g)^* \ast f$ has compact support disjoint from $\Xi$.

The proof that every $S_p$-set is an $S_q$-set is similar.

2. Examples of $S_p$- and $C_p$-sets.

Theorem 2.1. For $p \in [2, \infty]$ every closed subset of $\Gamma$ is a $C_p$-set.

Proof. In view of Theorem 1.5 we need only prove the theorem for $p = 2$.

Let $\Xi$ be a closed subset of $\Gamma$ and suppose we are given $\epsilon > 0$ and $f \in L^1 \cap C_0(G)$ with $\text{supp}(\hat{f})$ compact, $\hat{f}$ vanishing on $\Xi$ and $\|f\|_1 \leq 1$. Now $\Omega = \{\gamma \in \Gamma: \hat{f}(\gamma) \neq 0\}$ is a relatively compact open set, and hence there exists a compact set $Y \subset \Omega$ such that $\theta(\Omega \setminus Y) < \epsilon^2$. Choose an open set $\mathcal{V}$ such that $Y \subset \mathcal{V} \subset \mathcal{V} \cap \Omega$, and (see [3], 2.6.1) $k \in L^1 \cap C_0(G)$ such that $\xi_Y \leq \hat{k} \leq \xi_v$. Then, using Plancherel's theorem,

$$\|f - k \ast f\|_b = \left(\int_{\Omega \setminus Y} \left|1 - \hat{k}(\gamma)\right|^2 |\hat{f}(\gamma)|^2 d\theta(\gamma)\right)^{1/2} < \theta(\Omega \setminus Y)^{1/2} < \epsilon;$$

and clearly, $\hat{k}$ has compact support disjoint from $\Xi$.

Definition 2.2. Let $\Omega$ be a relatively compact open subset of $\Gamma$. We shall call $\Omega$ a $\beta$-symmetry set ($\beta > 0$) if there exist nets $\{Y_i\}_{i \in I}$ and $\{\mathcal{V}_i\}_{i \in I}$ such that each $Y_i$ is compact, $\{\mathcal{V}_i\}_{i \in I}$ is a base of symmetric open neighbourhoods of zero in $\Gamma$, partially ordered by $\mathcal{V}_i < \mathcal{V}_j$ if and only if $\mathcal{V}_i \supset \mathcal{V}_j$, $(Y_i + 2\mathcal{V}_i) \subset \Omega$ for each $i \in I$, and

$$\lim_{i \in I} \frac{\theta(\Omega \setminus Y_i)^{\beta}}{\theta(\mathcal{V}_i)} = 0.$$
Theorem 2.3. Suppose we are given $\beta > 0$ and a closed subset $\Xi$ of $\Gamma$ with the property that for any relatively compact set $Y \subset \Xi$ there exists a $\beta$-symmetry set $\Omega$ such that $Y \subset \Omega \subset \Xi$. Then $\Xi$ is a $C_p$-set for all $p \geq (2 + \beta)^{-1}(2 + 2\beta)$.

Proof. Let $p = (2 + \beta)^{-1}(2 + 2\beta)$. Suppose we are given $\epsilon > 0$ and $f \in L^1 \cap C_0(G)$, where $\text{supp}(\hat{f})$ is compact, $\hat{f}$ vanishes on $\Xi$ and $\|f\|_p \leq 1$. Now $Y = \{\gamma \in \Gamma : \hat{f}(\gamma) \neq 0\}$ is a relatively compact open subset of $\Xi$ and hence, by assumption, there exists a relatively compact open set $\Omega$ such that $Y \subset \Omega \subset \Xi$, and nets $\{Y_i\}_{i \in I}$ and $\{\nabla_i\}_{i \in I}$ satisfying the conditions of Definition 2.2. Choose $i \in I$ such that $Y_i$ is nonvoid and

$$\left[ \frac{\theta(\Omega \setminus Y_i)^{\beta}}{\theta(\nabla_i)} \right]^{\alpha/2} < 2^{-\alpha}\theta(\Omega)^{-\alpha/2}\epsilon,$$

where $\alpha = (1 + \beta)^{-1}$. Define $k_i = \theta(\nabla_i)^{-1}g_i, h_i$, where $g_i, h_i$ in $L^2(G)$ are such that $\hat{g}_i = \xi_{\nabla_i}$ (cf. [3], 2.6.1) $k_i \in L^1 \cap C_0(G)$, $\xi_{\nabla_i} \equiv \hat{k}_i \equiv \xi_{\nabla_i}^{1/2}$, and

$$\|k_i\|_p \leq \left[ \frac{\theta(\Omega + \nabla_i)}{\theta(\nabla_i)} \right]^\frac{1}{2}.$$

It follows from Hölder’s inequality that

$$\|f - k_i * f\|_p \leq \|f - k_i * f\|_p \|f - k_i * f\|_2^{-\alpha} \leq \|f\|_p \left[ 1 + \left[ \frac{\theta(\Omega + \nabla_i)}{\theta(\nabla_i)} \right]^{\frac{1}{2}} \right]^{\alpha} \theta(\Omega | Y_i)^{(1 - \alpha)/2} \leq 2^\alpha \theta(\Omega + \nabla_i)^{\alpha/2} \theta(\Omega | Y_i)^{(1 - \alpha)/2} \theta(\nabla_i)^{\alpha/2} \sim \epsilon$$

(recall that $\alpha = (1 + \beta)^{-1}$ and $p = (2 + \beta)^{-1}(2 + 2\beta) = 2(1 + \alpha^{-1})^{-1}$). Noting that $\hat{k}_i$ has compact support disjoint from $\Xi$ we see that $\Xi$ is a $C_p$-set, and the conclusion follows from Theorem 1.5.

We have two corollaries when $G$ is a Euclidean space.

Corollary 2.4. Let $m \geq 1$ and suppose $\Xi \subset \mathbb{R}^m$ is an open set with the property that for any relatively compact set $Y \subset \mathbb{R}^m$ there exists a number $\kappa_m (= \kappa_m(Y))$ such that

$$\theta((\partial(\Xi) \cap Y) + \nabla_n) \leq \kappa_m n^{-1}$$
for all \( n \in \{1, 2, \cdots \} \), where \( \partial (\Xi) \) denotes the boundary of \( \Xi \) and

\[
\nabla_n = \{ x \in \mathbb{R}^m : \| x \| < n^{-1} \}.
\]

Then \( \Xi, \Xi^c \) and \( \partial (\Xi) \) are \( C_p \)-sets for all \( p > (2 + m)^{-1}(2 + 2m) \).

**Proof.** By Theorem 1.3 (c) we need consider only \( \partial (\Xi) \).

Let \( Y \) be any relatively compact open subset of \( \partial (\Xi)^c \). We shall show that for any \( \epsilon > 0 \) there exists an \( (m + \epsilon) \)-symmetry set \( \Omega \) such that \( Y \subseteq \Omega \subseteq \partial (\Xi)^c \). Since \( Y \) is relatively compact in \( \mathbb{R}^m \) there exists an integer \( n_0 > 0 \) such that

\[
Y \subseteq \Delta_{n_0} = \{ x \in \mathbb{R}^m : \| x \| < n_0 \}.
\]

For each \( n \in \{1, 2, \cdots \} \) define

\[
Y_n = (\partial (\Xi) + \nabla_n)^c \cap (\Delta_{n_0} \Delta_{n_0 - n^{-1}})^c \cap \Delta_{n_0}.
\]

Clearly \( Y_n \) is compact and

\[
(Y_n + 2\nabla_{3n})^c \subseteq \Delta_{n_0} \cap \partial (\Xi)^c.
\]

Putting \( \Omega = \Delta_{n_0} \cap \partial (\Xi)^c \) we have

\[
\Omega \setminus Y_n = (\Omega \cap (\partial (\Xi) + \nabla_n)) \cup (\Omega \cap (\Delta_{n_0} \Delta_{n_0 - n^{-1}}))
\]

\[
= (\Delta_{n_0} \cap \partial (\Xi)^c \cap (\partial (\Xi) + \nabla_n)) \cup (\Delta_{n_0} \cap \partial (\Xi)^c \cap (\Delta_{n_0} \Delta_{n_0 - n^{-1}}))
\]

\[
\subseteq (\Delta_{n_0} \cap (\partial (\Xi) + \nabla_n)) \cup (\Delta_{n_0} \Delta_{n_0 - n^{-1}})
\]

\[
\subseteq (((\Delta_{n_0} + \nabla_n) \cap \partial (\Xi)) + \nabla_n) \cup (\Delta_{n_0} \Delta_{n_0 - n^{-1}}).
\]

Hence, since \( \Delta_{n_0} + \nabla_1 \) is relatively compact,

\[
\theta (\Omega \setminus Y_n) \leq \kappa_m (\Delta_{n_0} + \nabla_1) n^{-1} + O (n^{-1}).
\]

Using the fact that

\[
\theta (\nabla_{3n}) = \kappa'_m 3^{-m} n^{-m}
\]

for some constant \( \kappa'_m \), we have

\[
\lim_{n \to \infty} \frac{\theta (\Omega \setminus Y_n)^{m+\epsilon}}{\theta (\nabla_{3n})} = 0,
\]

and so \( \Omega \) is an \( (m + \epsilon) \)-symmetry set for all \( \epsilon > 0 \).
Thus $\partial(\Xi)$ satisfies the conditions of Theorem 2.3 with $\beta = m + \epsilon$, and hence is a $C_p$-set for all $p > (2 + m)^{-1}(2 + 2m)$.

**Corollary 2.5.** Let $m \geq 1$ and put

$$\Xi = \{x \in R^m : \|x\| = 1\}.$$ Then $\Xi$ is a $C_p$-set for all $p > (2 + m)^{-1}(2 + 2m)$.

*Proof.* Let $\nabla$ be any relatively compact set in $R^m$. Then

$$\theta((\Xi \cap \nabla) + \nabla_n) \leq \theta(\Xi + \nabla_n)$$

$$= \kappa'_m((1 + n^{-1})^m - (1 - n^{-1})^m)$$

$$= O(n^{-1}),$$

where $\kappa'_m$ is a constant. Now apply Corollary 2.4.

**Remark 2.6.** For $m \geq 3$, Corollary 2.5 gives an example of a $C_p$-set $((2 + m)^{-1}(2 + 2m) < p < 2)$ which is not an $S$-set; cf. [3], 7.3.2.

3. **The Failure of Certain Closed Sets to be $S_p$-Sets.** In this section we use a proof along the lines of that of Malliavin's theorem ([3], 7.6.1) to show that every nondiscrete $\Gamma$ contains a closed set which is not an $S_p$-set for any $p \in [1, 2)$. As in the proof of [3], Theorem 7.6.1, we first consider the cases:

(a) $\Gamma$ is an infinite compact group;

(b) $\Gamma = R$.

**Theorem 3.1.** Let $G$ be an infinite discrete group. Then there exists a closed set $\Xi \subset \Gamma$ which is not an $S_p$-set for any $p \in [1, 2)$.

*Proof.* Using the notation of [3], Theorem 7.8.6 we consider the function $\phi_1$ on $G$ defined by

$$\phi_1 : x \mapsto (D^1 m_x)(\xi).$$

It is easily proved from [3], 7.6.4 and Theorem 7.8.6 that $f_0 \in L^1(G)$ and $\phi_1$ (as above) can be chosen so that $f_0$ and $\xi$ satisfy the hypotheses of [3], 7.6.3 (Theorem) (with $f = f_0$ and $\xi = \xi$) and $\phi_1 \in L^q(G)$ for all $q > 2$. Having thus chosen $f_0$ and $\phi_1$ we shall prove that the closed set $\Xi = \{\gamma \in \Gamma : f_0(\gamma) = \xi\}$ is not an $S_p$-set for any $p \in [1, 2)$. 

Let $p \in [1, 2)$ and put

$$I = \{ f \in L'(G) : \hat{f}(\Xi) = \{0\} \},$$

$I_1$ the closed ideal of $L'(G)$ generated by $f_0 - \xi \xi_{[0]}$, 

$I_2$ the closed ideal of $L'(G)$ generated by $(f_0 - \xi \xi_{[0]})^{*2}$, 

and $J = \{ f \in L'(G) : \hat{f} \text{ vanishes on a neighbourhood of } \Xi \}$. 

Clearly

$$\Xi = Z(I) = Z(I_1) = Z(I_2) = Z(J)$$

(where $Z(I)$ denotes the zero set of the ideal $I$; see [3], 7.1.3). Since $I$ and $J$ are respectively the largest and smallest closed ideals in $L'(G)$ having $\Xi$ as their zero set, we have that $J \subset I_2 \subset I_1 \subset I$.

As $\phi_i \in L^p(G)$ we can define a continuous linear functional $T$ on $(L'(G), \| \cdot \|_p)$ by

$$T(g) = \sum_{x \in G} g(-x)\phi_i(x)$$

(recall that $G$ is discrete and hence $L'(G) \subset L^p(G)$). By [3], 7.6.3, $T$ annihilates $I$, but not $I_1$.

Now suppose that $\Xi$ is an $S_p$-set and let $h \in L^1 \cap C_0(G) = L'(G)$ with $\hat{h}$ vanishing on $\Xi$. Then, given $\epsilon > 0$, there exists $h' \in J$ such that

$$\| h - h' \|_p < \epsilon$$

and hence, since $T(h') = 0$, $|T(h)| = |T(h - h')| \leq \epsilon \| \phi_i \|_p$. As this holds for all $\epsilon > 0$ we must have that $T(h) = 0$; thus $T$ annihilates $I$, a contradiction of the fact that $T$ does not annihilate $I_1 \subset I$. It follows that $\Xi$ is not an $S_p$-set for any $p \in [1, 2)$.

We shall now examine the case when $\Gamma$ contains an infinite compact open subgroup. We require two lemmas for arbitrary Hausdorff locally compact Abelian groups.

**Lemma 3.2.** Let $G$ be a Hausdorff locally compact Abelian group and suppose $H$ is a closed subgroup of $G$. Then a continuous integrable function $f$ on $G$ is constant on cosets of $H$ if and only if

$$\text{supp}(\hat{f}) \subset A(\Gamma, H)$$

(the annihilator of $H$ in $\Gamma$).

**Proof.** The result follows readily from the property

$$(\gamma f)(\gamma) = \gamma(h) \hat{f}(\gamma)$$

for all $\gamma \in \Gamma$ (where $\gamma f : x \mapsto f(x + h)$).
Lemma 3.3. Let $G$ be a Hausdorff locally compact Abelian group and suppose $\Lambda$ is an open subgroup of $\Gamma$. If $\Xi$ is a closed subset of $\Lambda$ which is not an $S_p$-set in $\Lambda$ then $\Xi$ is not an $S_p$-set in $\Gamma$.

Proof. Put $H = A(G, \Lambda)$. By [1], (23.24) (e), $H$ is compact. Furthermore, in view of Theorem 2.1, we can assume that $p < \infty$.

Suppose, to the contrary, that $\Xi$ is an $S_p$-set in $\Gamma$. Given $\epsilon > 0$ and $f \in L^1 \cap C_0(G/H)$ such that $\text{supp}(f)$ is compact and $\hat{f}$ vanishes on $\Xi$, put $f = \hat{f} \circ \pi_H$, where $\pi_H$ denotes the natural homomorphism of $G$ onto $G/H$. Denoting the Haar measures on $H$, $G/H$ by $\lambda_H$, $\lambda_{G/H}$ respectively (normalised as in [2], Chapter 3, 3.3 (i) with $\lambda_H(H) = 1$) we have, by [2], Chapter 3, 4.5,

$$\|f\|_p = \int_{G/H} \left\{ \int_H |f(y)|^p d\lambda_H(y) \right\} d\lambda_{G/H}(x)$$

$$= \int_{G/H} \left\{ \int_{H} |\hat{f} \circ \pi_H(x)|^p d\lambda_H(y) \right\} d\lambda_{G/H}(x)$$

$$= \int_{G/H} |\hat{f}(x)|^p d\lambda_{G/H}(x),$$

that is,

(3.1) $$\|f\|_p = \|\hat{f}\|_p.$$

It is easily seen that

$$\hat{f}(x) = \int_{H} f(x + y) d\lambda_H(y)$$

and, by [2], Chapter 4, 4.3 ((3.1) shows that $f \in L^1(G)$),

(3.2) $$\hat{f}(\gamma) = f(\gamma)$$

for all $\gamma \in \Lambda$. Furthermore, since $f$ is constant on cosets of $H$, Lemma 3.2 shows that $\text{supp}(f) \subseteq A(\Gamma, H) = \Lambda$. As $\text{supp}(\hat{f})$ is assumed to be compact it follows from (3.2) that $\text{supp}(\hat{f})$ is compact and hence (note that $f$ is continuous) we see that $f \in C_0(G)$.

Now $\hat{f}$ vanishes on $\Xi \cup \Lambda^c$ and, since by Theorem 1.4 (recall that $\Lambda^c$ is open and closed) $\Xi \cup \Lambda^c$ is an $S_p$-set, there exists $g \in L^1 \cap C_0(G)$ such that $\hat{g}$ has compact support disjoint from $\Xi \cup \Lambda^c$ and $\|f - g\|_p < \epsilon$. By Lemma 3.2 again $g$ is constant on cosets of $H$ and we have the existence of $\hat{g} \in L^1 \cap C_0(G/H)$ such that $\hat{g} = \hat{g} \circ \pi_H (\hat{g} \in C_0(G/H)$ since, by [2], Chapter 3, 1.8 (vii), $\hat{g}$ is continuous and by (3.2), $\hat{g}$ has compact support). From (3.1) $\|\hat{f} - \hat{g}\|_p < \epsilon$, and (3.2) shows that $\hat{g}$ vanishes on a
neighbourhood of $\Xi$. Hence $\Xi$ is shown to be an $S_p$-set in $\Lambda$, contrary to assumption.

**Corollary 3.4.** Let $G$ be a Hausdorff locally compact Abelian group, $\Gamma$ its character group. If $\Gamma$ contains an infinite compact open subgroup then there exists a closed subset of $\Gamma$ which is not an $S_p$-set for any $p \in [1, 2)$.

**Proof.** Combine Theorem 3.1 and Lemma 3.3.

Before considering the case $\Gamma = \mathbb{R}$ we need to extend the result in [3], Theorem 2.7.6.

**Theorem 3.5.** Suppose $f \in l'(\mathbb{Z})$, $\delta \in (0, \pi)$ and $\hat{f}(\exp(ix)) = 0$ for $x \in [\pi - \delta, \pi + \delta]$. Let $u$ be defined on $\mathbb{R}$ by

$$u(x) = \begin{cases} \hat{f}(\exp(ix)) & (|x| \leq \pi) \\ 0 & (|x| > \pi). \end{cases}$$

Then $u = \hat{g}$ for some $g \in L'(\mathbb{R})$. Moreover, given $p \in [1, \infty]$, there exists a positive number $\kappa_p = \kappa_p(\delta)$ such that

$$\|f\|_p \leq \kappa_p \|g\|_p.$$

**Proof.** The first part of Theorem 3.5 is proved in [3], 2.7.6. Let $p \in [1, \infty]$. Consider the linear operator $T$ from $L^1 \cap L^\alpha(\mathbb{R})$ to $l'(\mathbb{Z})$, defined by

$$(3.3) \quad (T(k))(n) = k * \hat{h}(n),$$

where $n \in \mathbb{Z}$, and $h \in L'(\mathbb{R})$ is defined as in [3], 2.7.6. The argument at the end of the proof of [3], 2.7.6 shows that there is a constant $\kappa_1 = \kappa_1(\delta)$ such that $\|T(k)\|_1 \leq \kappa_1 \|k\|_1$. It is clear from (3.3) that $\|T(k)\|_\alpha \leq \kappa_2 \|k\|_\alpha$, where $\kappa_2 = \|\hat{h}\|$. By the Riesz-Thorin convexity theorem $T$ is continuous as

$$(L^1 \cap L^\alpha(\mathbb{R}), \|\cdot\|_\alpha) \overset{T}{\rightarrow} (l'(\mathbb{Z}), \|\cdot\|_\alpha)$$

(recall that $l'(\mathbb{Z}) \subset l^\alpha(\mathbb{Z})$, where $\alpha \in (0, 1)$, $p_\alpha = (1 - \alpha)^{-1}$ and $\|T\|_\alpha \leq \kappa_1 \alpha \kappa_2$. In particular, choosing $\alpha \in [0, 1)$ such that $p_\alpha = p$ (and $\alpha = 1$ if $p = \infty$) and noting that $g \in L^1 \cap L^\alpha(\mathbb{R})$ (and see [3], 2.7.6, (5)) $f(n) = g * \hat{h}(n)$ for all $n \in \mathbb{Z}$, we have

$$\|f\|_p \leq \kappa_1^{-\alpha} \kappa_2 \|g\|_p,$$

as required.
Theorem 3.6. The real line $\mathbb{R}$ contains a closed set which is not an $S_p$-set for any $p \in [1, 2)$.

Proof. It appears from Theorem 3.1 that there exists a closed set $\Xi_1 \subset T$ (the circle group) which is not an $S_p$-set for any $p \in [1, 2)$. By translation if necessary we can assume that $-1 \not\in \Xi_1$ and that $\Xi_1$ is disjoint from $\Xi_2$ for some closed arc $\Xi_2 \subset T$ containing $-1$. Put

$$Y_1 = \{x \in (-\pi, \pi): \exp(ix) \in \Xi_1\},$$
$$Y_2 = \{x \in (-\pi, \pi): \exp(ix) \in \Xi_2\} \cup [\pi, \infty) \cup (-\infty, -\pi],$$
$$\Xi = \Xi_1 \cup \Xi_2 \text{ and } Y = Y_1 \cup Y_2.$$

Let $p \in [1, 2)$ and suppose $Y_1$ is an $S_p$-set. By Theorem 1.4, $Y$ is an $S_p$-set. Given $f \in l'(Z)$ with $\hat{f}((\Xi)) = \{0\}$ define $g \in L^1 \cap C_0(\mathbb{R})$ by

$$\hat{g}(x) = \begin{cases} \hat{f}(\exp(ix)) & (|x| \leq \pi) \\ 0 & (|x| > \pi) \end{cases}$$

(see Theorem 3.5). Clearly $\hat{g}$ vanishes on $Y$ and hence, since $Y$ is an $S_p$-set, there exists a sequence $(g_n) \subset L^1 \cap C_0(\mathbb{R})$ such that each $\hat{g}_n$ vanishes on a neighbourhood of $Y$ and

$$(3.4) \quad \|g - g_n\|_{p} \to 0.$$

If, for each $x \in (-\pi, \pi]$, we define $f_n \in l'(Z)$ by

$$\hat{f}_n(\exp(ix)) = \hat{g}_n(x)$$

(see [3], Theorem 2.7.6) then Theorem 3.5 applied to (3.4) gives $\|f - f_n\|_{p} \to 0$ (note that each $\hat{f}_n$ vanishes on a neighbourhood of $\Xi$). Hence $\Xi$ and consequently (see Theorem 1.4) $\Xi_1$ would be an $S_p$-set, contradicting our choice of $\Xi_1$. It follows that $Y_1$ is not an $S_p$-set for any $p \in [1, 2)$.

We require two lemmas before proving the main result of this section.

Lemma 3.7. Let $G, H$ be Hausdorff locally compact Abelian groups and suppose $k \in L^1 \cap C_0(G \times H)$ is such that $Y = \text{supp}(k)$ is compact. Then the function $y \to k(x, y)(x \to k(x, y))$ is integrable over...
H for every \( x \in G \) (over \( G \) for every \( y \in H \)). Furthermore the functions

\[
\phi_1: x \mapsto \int_H k(x, y) d\lambda_H(y), \quad \phi_2: y \mapsto \int_G k(x, y) d\lambda_G(x)
\]

are continuous.

\textbf{Proof.} Since \( k \) is continuous the function \( y \mapsto k(x, y) \) is continuous, and hence measurable, for every \( x \in G \).

Choose \( k_1, k_2 \) in \( L^1 \cap C_0(G)(L^1 \cap C_0(H)) \) such that \( k_1 = 1 \) (\( k_2 = 1 \)) on a neighbourhood \( \nabla_1(\nabla_2) \) of \( Y_G(Y_H) \), where \( Y_G, Y_H \) are the projections of \( Y \) onto \( G, H \) respectively. If we define \( h \) on \( G \times H \) by \( h[(x, y)] = k_1(x)k_2(y) \) then [1], (31.7) (b) shows that \( \hat{h} = 1 \) on \( \nabla_1 \times \nabla_2 \), a neighbourhood of \( Y \). Thus \( h \ast k = k \) 1.a.e. and, since \( h \ast k \) and \( k \) are continuous, \( h \ast k = k \).

Now the map \( \nu \) on \( H \times G \times H \), defined by

\[
\nu \left[ (y, s, t) \right] = h(x - s, y - t) k(s, t),
\]

is continuous for every \( x \in G \). Applying [1], (13.4) to \( |\nu| \), considered as a function on \( H \times (G \times H) \), it follows that \( \nu \) is integrable and, using (3.5), that the function \( y \mapsto k(x, y) \) is integrable over \( H \) for every \( x \in G \). Furthermore, since \( \nu \) is integrable on \( H \times (G \times H) \), we can use (3.5) and [1], (13.8) to deduce that

\[
\phi_1(x) = \int_H k_2(y) d\lambda_H(y) \int_{G \times H} k_1(x - s) k(s, t) d\lambda_G \times \lambda_H(s, t).
\]

As \( k \in L^1(G \times H) \), \( k_2 \in L^1(H) \) and \( k_1 \) is uniformly continuous it follows that \( \phi_1 \) is continuous.

The other part of the lemma is proved similarly.

\textbf{Lemma 3.8.} Suppose \( G, H \) are Hausdorff locally compact Abelian groups, with character groups \( \Gamma, \Lambda \) respectively. If \( p \in [1, 2) \) and the closed set \( \Xi' \subset \Gamma \) is not an \( S_p \)-set, then \( \Xi = \Xi' \times \Lambda \) is not an \( S_p \)-set in \( \Gamma \times \Lambda \).

\textbf{Proof.} Suppose to the contrary that \( \Xi \) is an \( S_p \)-set in \( \Gamma \times \Lambda \). Let \( f \in L^1 \cap C_0(G) \) with \( \text{supp}(\hat{f}) \) compact and \( \hat{f} \) vanishing on \( \Xi \), and choose \( g \in L^1 \cap C_0(H) \) such that \( \text{supp}(\hat{g}) \) is compact and \( |g(y)| \equiv 1 \) for all \( y \) in
some neighbourhood $V$ of zero in $H$. Define $h$ on $G \times H$ by $h[(x, y)] = f(x)g(y)$. Then, by [1], (31.7) (b), $\text{supp}(\hat{h})$ is compact and

$$\hat{h}([\gamma_1, \gamma_2]) = \hat{f}(\gamma_1)\hat{g}(\gamma_2) = 0$$

for all $[\gamma_1, \gamma_2] \in \Xi$.

Let $\epsilon > 0$ be given. Since $\Xi$ is assumed to be an $S_p$-set we can find $k \in L^1 \cap C_0(G \times H)$ such that $\text{supp}(\hat{k})$ is compact and disjoint from $\Xi$, and

$$\|h - k\|_p < \epsilon \lambda_H(V)^{1/p}.$$  

Thus, for all $\gamma_1$ in some neighbourhood $\nabla$ of $\Xi'$ and for all $\gamma_2 \in \Lambda$, we have (see [1], (13.8))

$$\int_H \left\{ \int_G k(x, y)\bar{\gamma}_1(x)d\lambda_G(x) \right\} \bar{\gamma}_2(y)d\lambda_H(y)$$

$$= \int_{G \times H} k(x, y)([\gamma_1, \gamma_2])^{-1}(x, y)d\lambda_G \times \lambda_H(x, y)$$

$$= 0.$$

Since $\gamma_2 \in \Lambda$ was chosen arbitrarily

$$\int_G k(x, y)\bar{\gamma}_1(x)d\lambda_G(x) = 0 \quad \lambda_H - \text{a.e.}$$

Now

$$\psi: (x, y) \rightarrow k(x, y)\bar{\gamma}_1(x)$$

is continuous and integrable, and $\text{supp}(\hat{\psi})$ is compact. Hence, by Lemma 3.7, the function $\phi$ on $H$ defined by

$$\phi(y) = \int_G \psi(x, y)d\lambda_G(x)$$

is continuous and so, for all $y \in H$ and $\gamma_1 \in \nabla$,

$$(3.7) \quad \int_G k(x, y)\bar{\gamma}_1(x)d\lambda_G(x) = 0.$$ 

Using (3.6) we see that

$$W = \left\{ y \in V: \int_G |h(x, y) - k(x, y)|^p d\lambda_G(x) < \epsilon^p \right\}$$
has the property that $\lambda_H(V \setminus W) < \lambda_H(V)$, that is, $\lambda_H(W) > 0$. Choose any $y_0 \in W$ ($W$ is nonempty). Then

$$
(3.8) \quad \int_G |f(x) - g(y_0)^{-1}k(x, y_0)|^p d\lambda_G(x) < \epsilon^p |g(y_0)|^{-1} \leq \epsilon^p
$$

and so, defining $f_1 \in L^1 \cap C_0(G)$ by $f_1(x) = g(y_0)^{-1}k(x, y_0)$, (3.7) shows that $\hat{f}_1$ vanishes on $\nabla$ and, from (3.8), $\|f - f_1\|_p < \epsilon$; thus we have a contradiction of the assumption that $\Xi'$ is not an $S_p$-set.

**Theorem 3.9.** Let $G$ be a Hausdorff noncompact locally compact Abelian group, $\Gamma$ its character group. Then $\Gamma$ contains a closed set which is not an $S_p$-set for any $p \in [1, 2)$.

**Proof.** By [1], (24.30), $\Gamma$ is topologically isomorphic with $\mathbb{R}^n \times \Gamma_0$, where $\Gamma_0$ is a Hausdorff locally compact Abelian group containing a compact open subgroup.

If $n \geq 1$ then Theorem 3.6 and Lemma 3.8 combine to show that $\mathbb{R}^n \times \Gamma_0$ contains a closed set which is not an $S_p$-set for any $p \in [1, 2)$.

If $n = 0$ then $\Gamma$ contains a compact open subgroup (with is infinite since $\Gamma$ is nondiscrete) and the result follows from Corollary 3.4.

**References**


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