STRICT LOCAL INCLUSION RESULTS BETWEEN
SPACES OF FOURIER TRANSFORMS

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Let \( G \) denote a noncompact Hausdorff locally compact abelian group, \( \Gamma \) its character group, and write \( (L^s, l^t) \)\(^{loc}\) for the space of Fourier transforms of functions in the amalgam \( (L^s, l^t) \). We show that for \( 1 \leq p < q \leq \infty \) the local inclusion \( (L^s, l^t) \)\(^{loc}\) \( \subset (L^{s'}, l^{t'}) \) is strict, that is, given any nonvoid open subset \( \Omega \) of \( \Gamma \) there exists \( f \in (L^s, l^t) \) such that \( \hat{f} - \hat{g} \) does not vanish on \( \Omega \) for any \( g \in (L^s, l^t) \). If in addition \( G \) is assumed to be second countable then we show there exists such an \( f \) independent of the choice of \( \Omega \). Of special interest is the case, included in the above results, where the amalgams \( (L^s, l^t) \), \( (L^{s'}, l^{t'}) \) are replaced by \( L^p(G) \), \( L^q(G) \) respectively.

Throughout \( G \) will denote a noncompact Hausdorff locally compact abelian group, with Haar measure \( \lambda \) and character group \( \Gamma \). \( A(\Gamma) \) will denote the space of Fourier transforms of functions integrable over \( G \), and \( A_c(\Gamma) \) the subspace formed of functions whose supports are compact. For each compact set \( E \subset \Gamma \), write \( A_s(\Gamma) = \{ h \in A(\Gamma) : \text{supp}(h) \subset E \} \). We give each space \( A_s(\Gamma) \) its normed topology as \( \| \hat{f} \| = \| f \|_s, \hat{f} \in A_s(\Gamma) \), and topologise \( A_s(\Gamma) \) as the internal inductive limit of the spaces \( A_s(\Gamma) \).

For each \( s, t \in [1, \infty] \) the amalgam \( (L^s, l^t) \) is defined in the following way. Using the structure theorem ([6], (24.30)) we write \( G = R^a \times G_0 \), where \( a \) is a nonnegative integer and \( G_0 \) contains a compact open subgroup \( H \). We put \( J = Z^a \times G_0/H, \) \( L = [0, 1)^a \times H \) and write \( G \) as the disjoint union \( \bigcup_a L_a \) where, for each \( \alpha = (n_1, \ldots, n_a, y + H) \in J, L_a = (n_1, \ldots, n_a, y) + L \). Given \( f \in L^s_{loc}(G) \) write

\[
\| f \|_{s,t} = \left( \sum_a \left( \int_{L_a} |f|^s d\lambda \right)^{t/s} \right)^{1/t},
\]

with the usual modification if \( \max \{ s, t \} = \infty \), and

\[
(L^s, l^t) = \{ f \in L^s_{loc}(G) : \| f \|_{s,t} < \infty \}.
\]

Each amalgam \( (L^s, l^t) \) is a Banach space and, provided \( s, t < \infty \), its dual space is isometrically isomorphic to \( (L^{s'}, l^{t'}) \) (where \( s', t' \) denote the indices conjugate to \( s, t \) respectively); for this and other results on amalgams see [9], § 3.

Here we give three results for amalgams, which will be referred to in the sequel.
Theorem A. The translation operators \( \tau_b \) defined by \( \tau_b f: x \mapsto f(x - b) \) are uniformly bounded on \( (L^s, l^t) \) for all \( s, t \in [1, \infty] \).

Theorem B. For any compact set \( \Xi \subset \Gamma \) there exists \( k \in (L^\infty, l^1) \) with \( \hat{k} = 1 \) on \( \Xi \).

Theorem C. Let \( f \in (L^{p_1}, l^{q_1}), g \in (L^{p_2}, l^{q_2}) \) and suppose that \( 1/r_i = 1/p_i + 1/q_i - 1 \geq 0 \) for \( i = 1, 2 \). Then \( f \ast g \in (L^r, l^s) \) and \( \| f \ast g \|_{r_i, r_2} \leq K \| f \|_{p_i, p_2} \| g \|_{q_1, q_2} \), where \( K \) is a constant.

The proof of Theorem 3.3 in [9] applies equally well to give Theorem A, and Theorem B is just [9], Theorem 3.1. Theorem C follows from Theorem A and the results of [1], §7(i).

For each \( f \in (L^l, l^\infty) \) we define the Fourier transform \( \hat{f} \) as the continuous linear functional on \( A_\infty(\Gamma) \) given by

\[
\hat{f}(h) = f(\hat{h}_\nu), \quad h \in A_\infty(\Gamma),
\]

where \( \hat{h}_\nu \) is the reflection of the inverse Fourier transform of \( h \) and \( f(g) = \int G f g d\nu \) (for a similar definition of the Fourier transform see Bertrandias and Dupuis ([2], §4(a))). That \( \hat{f} \) is linear is clear, and continuity can be shown as follows. Since \( A_\infty(\Gamma) \) is the inductive limit of the spaces \( A_s(\Gamma) \) we need only prove that \( \hat{f} \) is continuous on each \( A_s(\Gamma) \). First note that if \( h \in A_s(\Gamma) \) then \( \hat{h}_\nu = k \ast \hat{h}_\nu \in (L^\infty, l^1) \) by Theorem C, where \( k \in (L^\infty, l^1) \) is chosen as in Theorem B with \( \hat{k} = 1 \) on \(-\Xi\). Hence

\[
|\hat{f}(h)| = |f(\hat{h}_\nu)| = \left| \int G f \hat{h}_\nu d\lambda \right| \leq \sum_a \int_{L_a} |f \hat{h}_\nu| d\lambda \leq \sum_a \left( \int_{L_a} |f| d\lambda \max_{L_a} |\hat{h}_\nu| \right) \\
\leq \left( \max_a \int_{L_a} |f| d\lambda \right) \sum_a \max_{L_a} |\hat{h}_\nu| \\
= \| f \|_{1, \infty} \| \hat{h}_\nu \|_{\infty, 1} \\
\leq K \| f \|_{1, \infty} \| k \|_{\infty, 1} \| \hat{h}_\nu \|_1 \\
= K \| f \|_{1, \infty} \| k \|_{\infty, 1} \| h \|,
\]

so that \( \hat{f} \) is bounded, hence continuous, on \( A_\infty(\Gamma) \).

It follows from [3], Theorem 3.1 (and the remarks at the beginning of [3], §3) that this definition of the Fourier transform agrees with that given by Gaudry in [5], §1 (who defines the Fourier transform as a suitable quasimeasure) for functions in \( L^p(\Gamma) \), and in particular with that usually taken when \( 1 < p \leq 2 \). It will also be convenient to think of the Fourier transform of \( f \in L^l(\Gamma) \) as a
linear functional on $A_e(\Gamma)$. We write $(L^*, l^r) = \{\hat{f} : f \in (L^*, l^r)\}$.

Let $\Omega$ be a nonvoid open subset of $\Gamma$. We say that $\hat{f}$ vanishes on $\Omega$ if $\hat{f}(h) = 0$ for all $h \in A_e(\Gamma)$ with $\text{supp}(h) \subset \Omega$. Suppose that $1 \leq p \leq q \leq \infty$, $g \in (L^q, l^p)$, and take $\Omega$ to be any nonvoid relatively compact open subset of $\Gamma$. By Theorem B we have the existence of $k \in (L^\infty, l^p)$ with $k = 1$ on $\Omega$. Writing $f = k \ast g$ it is seen that $f \in (L^\infty, l^p)$ and $\hat{f} - \hat{g}$ vanishes on $\Omega$. Thus we have that $(L^1, l^p) \subset (L^\infty, l^p)$, where the inclusion is to be interpreted as holding locally. This has already been observed for the Lebesgue spaces when $G$ is the real line; see [8], Chapter VI, § 4.12.

Our main result (Theorem 1), which we prove with the aid of an extension to amalgams of a multiplier theorem of Hörmander, implies that this local inclusion is strict whenever $p < q$. In the case $q \in (1, 2]$, with the amalgams $(L^1, l^p)$, $(L^\infty, l^p)$ replaced by $L^\alpha(G)$, $L^\beta(G)$ respectively, this result has been given previously by Fournier ([4], Theorem 1) with $\Omega$ only required to be a set with positive Haar measure. Fournier's proof is based on the construction of certain positive definite functions on $\Gamma$.

**Theorem 1.** Let $1 < q \leq \infty$ and suppose $\Omega$ is a nonvoid open subset of $\Gamma$. Then there exists $f \in (L^\infty, l^p)$ such that for any $p \in [1, q)$ and $g \in (L^1, l^p)$, $f - g$ does not vanish on $\Omega$.

**Proof.** We first show that if $h \in L^\alpha(G)$ with $h \geq 0$, $h \neq 0$, then there exists nonnegative $f \in (L^\infty, l^p)$ such that $h \ast f \in (L^1, l^p)$ for any $p \in [1, q)$. This is easy to see if $q = \infty$, since in this case $f = 1$ satisfies the stated conditions.

For $q < \infty$ fix $p \in [1, q)$ and assume that $h \ast f \in (L^1, l^p)$ for all $f \in (L^\infty, l^p)$. Consider the map $T$, defined on $(L^\infty, l^p)$ by $Tf = h \ast f$. By assumption $T$ maps $(L^\infty, l^p)$ into $(L^1, l^p)$. $T$ is obviously linear and commutes with translations. Furthermore the Closed Graph Theorem shows that $T$ is continuous. Now the proof of Hörmander's theorem ([7], Theorem 1.1), which holds for all noncompact locally compact abelian groups, can be modified to show that $T = 0$. This is clearly impossible if $h$ is nonzero, so there exists $f \in (L^\infty, l^p)$ such that $h \ast f \in (L^1, l^p)$. Since $h \geq 0$ the same is true if $f$ is replaced by $|f|$. Now let $(p_n)$, $p_n \geq 1$, be any strictly increasing sequence of numbers converging to $q$. Choose a corresponding sequence $(f_n)$ of nonnegative functions in $(L^\infty, l^p)$ such that for each $n \in \{1, 2, \ldots\}$, $h \ast f_n \in (L^1, l^{p_n})$. We assert that

$$f = \sum_{n=1}^{\infty} n^{-2} \|f_n\|_{l^{p_n}}^{-1} f_n$$
is a suitable choice of \( f \); indeed, if there exists \( p \in [1, q) \) such that
\[ h \ast f \in (L^1, l^p) \]
then, choosing \( n_0 \) such that \( p_{n_0} \in [p, q) \), we would have
(recall that for each \( n \), \( h \ast f_n \geq 0 \))
\[ h \ast f_{n_0} \in (L^1, l^p) \subset (L^1, l^{\infty}) , \]
contradicting the choice of \( f_{n_0} \).

Now choose \( \gamma \in \Omega \) and nonzero \( h \in \Lambda_c(\Gamma) \) such that \( \text{supp}(h) \subset -\gamma + \Omega \) and \( \hat{h} \geq 0 \) (this is possible using [6], (31.34) and the fact that
\(-\gamma + \Omega \) is a neighbourhood of \( 0 \)). From the first part of the proof
there exists \( f_0 \in (L^\infty, l^p) \) such that \( \hat{h} \ast f_0 \in (L^1, l^p) \) for any \( p \in [1, q) \). Then \( f = \gamma f_0 \)
satisfies the conditions of the theorem, for if there exists \( p \in [1, q) \) and \( g \in (L^1, l^p) \) such that \( f - g \) vanishes on \( \Omega \) then, since \( (\gamma f - \gamma g) \) vanishes on \(-\gamma + \Omega \), we would have
\[ \hat{h} \ast (\gamma f - \gamma g)(x) = (\gamma f - \gamma g)(xh) = 0 \]
for all \( x \in G \) (where \( (xh)(\gamma) = \gamma(x)h(\gamma) \)). But this gives \( \hat{h} \ast f_0 = \hat{h} \ast \gamma g \in (L^1, l^p) \), a contradiction of our choice of \( f_0 \).

In the case where \( G \) is second countable \( f \) in Theorem 1 can be
chosen independently of the nonvoid open set \( \Omega \).

**Theorem 2.** Let \( G \) be a second countable noncompact locally
compact abelian group. If \( 1 \leq p < q \leq \infty \) then there exists \( f \in (L^\infty, l^p) \)
such that, for any nonvoid open set \( \Omega \subset \Gamma \), there is no \( g \in (L^1, l^p) \)
for which \( f - g \) vanishes on \( \Omega \).

**Proof.** Since \( G \) is second countable so is \( \Gamma \) (see [6], (24.14)).
Suppose to the contrary that no such \( f \) exists when \( g \) is restricted
to lie in \( L^p(G) \). We consider \( p > 1 \) and make use of Baire's category
theorem to derive a contradiction.

For each pair of positive integers \( m, n \) define \( T_m(\Omega_n) = \{ f \in (L^\infty, l^p) : \hat{f} - \hat{g} \) vanishes on \( \Omega_n \) for some \( g \in L^p(G) \), \( \| g \|_p \leq m \} \), where
\( \{ \Omega_n : n = 1, 2, \ldots \} \) is a base for the topology of \( \Gamma \) with each \( \Omega_n \)
nonvoid. Our assumption in the previous paragraph just says that
\( \bigcup_{m,n} T_m(\Omega_n) = (L^\infty, l^p) \). We shall show that for each \( m, n \in \{1, 2, \ldots \} \),
\( T_n(\Omega_n) \) is closed.

Let \( (f_s) \) be a sequence of functions in \( T_m(\Omega_n) \) converging in
\( (L^\infty, l^p) \) to \( f \), say. Now for each \( s \in \{1, 2, \ldots \} \) there exists \( g_s \in L^p(G) \)
such that \( \| g_s \|_p \leq m \) and \( \hat{f}_s - \hat{g}_s \) vanishes on \( \Omega_n \). Using the theorem
of Alaoglu we can deduce the existence of \( g \in L^p(G) \) with \( \| g \|_p \leq m \),
a weak*-cluster point of the sequence \( (g_s) \).
Now let $\varepsilon > 0$ and $h \in A_s(I)$ with $\text{supp}(h) \subseteq \Omega_s$ be given. Choose $s$ such that $|g(\hat{h}_\nu) - g_s(\hat{h}_\nu)| < \varepsilon/2$ and $|f(\hat{h}_\nu) - f_s(\hat{h}_\nu)| < \varepsilon/2$ (note that $\hat{h}_\nu \in (L^\omega, l^\nu)$). Then

$$
|(\hat{f} - \hat{g})(h)| \leq |f(\hat{h}_\nu) - f_s(\hat{h}_\nu)| + |f_s(\hat{h}_\nu) - g_s(\hat{h}_\nu)|
+ |g_s(\hat{h}_\nu) - g(\hat{h}_\nu)| < \varepsilon + |(\hat{f}_s - \hat{g}_s)(h)| .
$$

But $\hat{f}_s - \hat{g}_s$ vanishes on $\Omega_s$ and thus $|(\hat{f} - \hat{g})(h)| < \varepsilon$. Since $\varepsilon > 0$ and $h \in A_s(I)$ with $\text{supp}(h) \subseteq \Omega_s$ were chosen arbitrarily we deduce that $\hat{f} - \hat{g}$ vanishes on $\Omega_s$, so that $f \in T_m(\Omega_s)$. Hence $T_m(\Omega_s)$ is closed.

Now $(L^\omega, l^\nu)$ is a complete metric space and thus we can apply Baire's category theorem which gives us the existence of positive integers $m_0, n_0$ such that $T_{m_0}(\Omega_{n_0})$ has nonvoid interior. This means we can find $\delta > 0$ and $f_0 \in T_{m_0}(\Omega_{n_0})$ such that

$$
V = \{ f \in (L^\omega, l^\nu) : \| f - f_0 \|_{\infty, q} < \delta \} \subseteq T_{m_0}(\Omega_{n_0}) .
$$

Let $k \in (L^\omega, l^\nu)$ and choose nonzero $\alpha$ such that $\|\alpha k\|_{\infty, q} < \delta$. Then $f_0, \alpha k + f_0$ belong to $V$ and so there exist $g_0, g_1 \in L^\sigma(G)$ with $\hat{f}_0 - \hat{g}_0$ and $(\alpha k + f_0)^\sim - \hat{g}_1$ vanishing on $\Omega_{n_0}$. The linearity of the Fourier transform entails that

$$
\hat{k} - \alpha^{-1}((g_1 - g_0)^\sim) = \alpha^{-1}((\alpha k + f_0 - g_1)^\sim - (f_0 - g_0)^\sim)
$$

vanishes on $\Omega_{n_0}$. But $\alpha^{-1}(g_1 - g_0) \in L^\sigma(G)$; since $k \in (L^\omega, l^\nu)$ was chosen arbitrarily we have a contradiction of Theorem 1.

Hence our initial assumption was false and, for $p_0 \in (p, q)$ with $p, q$ as in the statement of the theorem, we have the existence of $f \in (L^\omega, l^\nu)$ such that, for any nonvoid open set $\Omega \subset I$, there is no $g \in L^{p_0}(G)$ for which $\hat{f} - \hat{g}$ vanishes on $\Omega$. Then the same is true with $L^{p_0}(G)$ replaced by $(L^\sigma, l^\nu)$. For suppose to the contrary that $\Omega \subset I$ and $g \in (L^\sigma, l^\nu)$ exist such that $\hat{f} - \hat{g}$ vanishes on $\Omega$; without loss of generality we may assume that $\Omega$ is relatively compact. Then, choosing $k \in (L^{p_0}, l^\nu)$ with $\hat{k} = 1$ on $\Omega$, we have that

$$
\hat{f} - (k * g)^\sim = (\hat{f} - \hat{g}) + (\hat{g} - (k * g)^\sim)
$$

vanishes on $\Omega$ and $k * g \in L^{p_0}(G)$, contradicting our choice of $f$. This completes the proof of the theorem. \[\square\]

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