ON OPTIMUM SUMMABLE GRAPHS

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Communicated by: G.S. Bloom
Received 06 February 2006; accepted 13 May 2006

Abstract

For a graph \( G \), let \( \sigma(G) \) and \( \delta(G) \) denote, respectively, its sum number and minimum degree. Trivially, \( \sigma(G) \geq \delta(G) \). A nontrivial connected graph \( G \) is called a \( k \)-optimum summable graph, where \( k \geq 1 \), if \( \sigma(G) = \delta(G) = k \). In this paper, we show that if \( G \) is a \( k \)-optimum summable graph of order \( n \), \( k \geq 3 \), then (1) \( n \geq 2k \); (2) the complete bipartite graph \( K_{k,n-k} \) is not a spanning subgraph of \( G \). We also describe new families of \( k \)-optimum summable graphs for \( k \geq 1 \).

Keywords: Sum graph, Sum number, Sum labeling, Minimum degree
Optimum summable graph, Unit graph.

2000 Mathematics Subject Classification: 05C
1. Introduction

All graphs considered here are finite simple graphs. For a graph $G$, $V(G)$ will denote its vertex set and $E(G)$ its edge set, while $n(G)$ and $e(G)$ respectively denote the order and size of $G$; that is, $n = n(G) = |V(G)|$ and $e(G) = |E(G)|$. A graph $G$ is nontrivial if $n(G) \geq 2$. For other standard notation and terminology not explained here, refer to [1].

Let $N$ denote the set of positive integers. Following Harary [2], the sum graph $G^+(S)$ of a finite subset $S \subset N$ is the graph with vertex set $S$ and edge set $E$ such that for distinct $u, v \in S$, $uv \in E$ if and only if $u + v \in S$. By extension a graph $G$ is called a sum graph if it is isomorphic to the sum graph $G^+(S)$ of $S \subset N$.

The notion of sum graph can be defined equivalently as follows. For a graph $G$ with minimum degree $\delta(G) \geq 1$ and a positive integer $k$, we write $G_k$ for $G \cup \overline{K_k}$, the disjoint union of $G$ and $k$ isolated vertices. Then the graph $G_k$ is a sum graph if there exists an injective labeling $L : V(G_k) \rightarrow N$ such that for any two distinct vertices $u, v$ of $G_k$, $uv \in E(G_k)$ iff there exists $w \in V(G_k)$ with $L(w) = L(u) + L(v)$. In this case, $L$ is called a sum labeling of $G_k$. Observe that, by definition, the vertex with the largest label in a sum graph cannot be adjacent to any other vertex. Thus, if $G_k$ is a sum graph, then $k \geq 1$. For a connected graph $G$, its sum number, denoted by $\sigma(G)$, is defined as the smallest $k$ for which $G_k$ is a sum graph. Since the vertex with the largest label in $G$ is adjacent to at least $\delta(G)$ vertices, we have $\sigma(G) \geq \delta(G)$. Motivated by this relation, we define a nontrivial connected graph $G$ to be $k$-optimum summable, where $k \geq 1$, if $\sigma(G) = \delta(G) = k$. Following Harary [2], a nontrivial connected graph $G$ is called a unit graph if $G_1$ is a sum graph. Thus, $G$ is a unit graph iff it is 1-optimum summable.

The problem of characterizing $k$-optimum summable graphs (even when $k = 1$) is believed to be very difficult. In this paper, we shall first show in the next section that if $G$ is a $k$-optimum summable graph of order $n$, $k \geq 3$, then (1) $n \geq 2k$; (2) the complete bipartite graph $K_{k,n-k}$ is not a spanning subgraph of $G$. In the remaining sections we describe new families of $k$-optimum summable graphs for $k \geq 1$.

2. Necessary Conditions

Let $K_n$ denote the complete graph of order $n$. We have $\sigma(K_2) = 1$, $\sigma(K_3) = 2$ and so $K_2$ is 1-optimum summable and $K_3$ is 2-optimum summable. However, it is known [3] that $\sigma(K_n) = 2n - 3$ for $n > 4$, and therefore $K_n$ is not $(n-1)$-optimum summable.

For the rest of this paper, let $G$ be a $k$-optimum summable graph. Let $L$ be a sum labeling of $G_k$. For convenience, throughout this paper, we shall refer to the vertices of $G_k$ by their sum labels.

Let $u$ be the largest vertex in $V(G)$. Since $G$ is a $k$-optimum summable graph, we have $\deg(u) \geq k$. But since $u$ is the vertex with the largest label, $\deg(u) \leq k$, and so $\deg(u) = k$. Denoting by $N(x)$ the set of vertices adjacent to a given vertex $x$, let
Lemma 2.1. \( a_i + a_j \notin A \) for \( 1 \leq i < j \leq k \).

Proof. Suppose that there exist \( i, j \) with \( 1 \leq i < j \leq k \) such that \( a_i + a_j \in A \). Then \( k \geq 3 \) and \( a_i + a_j = a_p \) for some \( p \in j + 1..k \). As \( u + a_p \in V(G_k) \), \( u + a_i \) is adjacent to \( a_j \), contradicting the fact that \( u + a_i \) is an isolated vertex.

Lemma 2.2. \( b_i + a_j \notin A \) for every \( 1 \leq i \leq n - k - 1 \) and \( 1 \leq j \leq k \).

Proof. Suppose that \( b_i + a_j \in A \) for some \( j \in j + 1..k \). Then \( k \geq 2 \) and \( u + b_i + a_j \in V(G_k) \). Hence \( u + a_j \) is adjacent to \( b_i \), a contradiction.

Now let \( X = N(a_1) \setminus \{u\} = \{x_1, x_2, \ldots, x_{k'-1}\} \), where \( x_1 < x_2 < \cdots < x_{k'-1} \) and \( k' \geq k \). Obviously, \( X \subset A \cup B \).

Lemma 2.3. \( x_i + a_1 \notin C \) for every \( i \in 1..k' - 1 \).

Proof. Obvious since \( x_i + a_1 < u + a_1 \) for \( 1 \leq i \leq k' - 1 \).

Recall that for \( k \geq 3 \) a \( k \)-optimum summable graph \( G \) cannot be a complete graph, and so \( n(G) \geq \delta(G) + 2 \). However, as the next theorem shows, we can find a much better general lower bound on the order of a \( k \)-optimum summable graph.

Theorem 2.1. If \( G \) is a \( k \)-optimum summable graph for \( k \geq 3 \), then \( n(G) \geq 2k \).

Proof. Let \( G \) be a \( k \)-optimum summable graph with \( V(G) = \{u\} \cup A \cup B \) and \( V(G_k) = V(G) \cup C \) as described above.

Consider the edges between \( a_1 \) and its neighbours \( x_i, i = 1, \ldots, k' - 1 \), other than \( u \). By Lemma 2.1 and Lemma 2.2, \( a_1 + x_i \notin A \) for every \( i \in 1..k' - 1 \); by Lemma 2.3, \( a_1 + x_i \notin C \) for every \( i \in 1..k' - 1 \). Hence, for every \( i \in 1..k'-1 \), \( a_1 + x_i \in B \cup \{u\} \). Since \( a_1 \) is also adjacent to \( u \), this tells us that \( \deg(a_1) = k \leq |B| + 2 \), hence that \( |B| \geq k - 2 \). Since \( |B| = n - k - 1 \), it follows that \( n \geq 2k - 1 \).

Next we show that \( |B| \neq k - 2 \), thus proving that \( n \geq 2k \). If on the contrary we suppose that \( |B| = k - 2 \), then

(1) Every \( a_i \in A \) is adjacent to at least one other \( a_j \in A \).

(2) Every \( a_i \in A \) is adjacent to some \( x \neq u \) such that \( a_i + x \notin B \).
(3) If \( u = a_i + x \) for some \( x \in A \cup B \), then by Lemma 2.3 for every \( i' \in i .. k \), 
\((a_{i'}, x) \notin E\).

The edges involving \( a_1 \) can only sum to \( b_1, b_2, \ldots, b_{k-2}, u \) or \( c_1 = u + a_1 \) which implies that \( \deg(a_1) \) is at most \( k \), hence exactly \( k \). Thus there exists some \( x \in A \cup B \) such that \((a_1, x) \in E(G)\) and \( a_1 + x = u \). Two cases then arise, depending on whether \( x \in A \) or \( x \in B \):

Case 1 \( x \in A \)

Suppose \( x = a_j \) for some \( j \in 2 .. k \). Denoting by \( x_i, 1 \leq i \leq k \), the vertices adjacent to \( a_1 \) in ascending order, and recalling that the vertices of \( A \) and \( B \) are also listed in ascending order, we must have

\[ a_1 + x_1 = b_1, a_1 + x_2 = b_2, \ldots, a_1 + x_{k-2} = b_{k-2}, a_1 + a_j = u, a_1 + u = c_1, \]

where \( x_{k-1} = a_j \) and \( x_k = u \). Thus for some \( m \geq 2 \) we may arrange the vertices in ascending sequence as follows:

\[ a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_{k-2}, a_j. \]

Now consider \( a_j \). From (3) we know that for every \( j' > 1 \), \((a_{j'}, a_j) \notin E\). Thus \( a_j \) can be adjacent only to \( a_1, b_1, b_2, \ldots, b_{k-2} \) and \( u \), where \( a_j + a_1 = u \); therefore, for \( y \in B \cup \{u\} \), \( a_j + y \in C \). Since \( a_j + u = c_j \), it follows that \( j = k - 1 \) or \( k \).

(a) Suppose \( j = k - 1 \).

Here for every \( i \in 1 .. k - 2 \),

\[ c_i = b_i + a_{k-1} = a_i + u = (a_i + a_1) + a_{k-1}, \]

from which \( b_i = a_1 + a_i \). Thus \( a_1 \) is adjacent to \( a_2, a_3, \ldots, a_{k-2} \) as well as to \( a_{k-1} \) and \( u \), but by (3) not to \( a_k \). Hence \( a_1 \) must be adjacent to one vertex, say \( b_r \), in \( B \), and further, by Lemmas 2.1–2.3, \( a_1 + b_r = b_q \) for some \( q \in r + 1 .. k - 2 \).

At the same time \( b_q = a_1 + a_s \) for some \( a_s \) so that \( a_s = b_r \), giving duplicate labels in \( G \). Therefore \( j \neq k - 1 \).

(b) Suppose \( j = k \).

We conclude as in (a) that \( a_1 \) is adjacent to \( a_2, a_3, \ldots, a_{k-2} \), and in addition to \( a_k \) and \( u \). Suppose that \((a_1, a_{k-1}) \in E(G)\). But then \( a_1 + a_k \in B \), as in (a) an impossibility since \( b_i = a_1 + a_i \) for every \( i \in 1 .. k - 2 \). Thus \( j \neq k \).

We have shown that Case 1 is impossible.
Case 2  $x \in B$

Suppose $x = b_j$ for some $j \in 1..k-2$. Then $u = a_1 + b_j$, so that for every $i \in 1..k$, $c_i = a_1 + (a_i + b_j)$. Since $a_i + b_j > u$ for every $i > 1$, it follows that vertices $a_i + b_j$ cannot exist. Thus $b_j$ is not adjacent to any of $a_2, a_3, \ldots, a_k$, and so has degree at most $k - 2$, contradicting the requirement that $\delta = k$. Thus $u \neq a_1 + b_j$ and Case 2 is impossible.

On the assumption that $|B| \leq k - 2$, we have shown that $a_1 + x \neq u$ for any $x$. Hence $|B| \geq k - 1$, as required. \qed

The next result gives us more insight into the structure of a $k$-optimum summable graph.

**Theorem 2.2.** If $G$ is a $k$-optimum summable graph, $k \geq 3$, then $K_{k,n-k}$ is not a spanning subgraph of $G$.

**Proof.** Suppose to the contrary that there exists a $k$-optimum summable graph $G$ such that $G$ contains $K_{k,n-k}$ as a spanning subgraph. As before, let $V(G) = \{u\} \cup A \cup B$ and $V(G_k) = \{u\} \cup A \cup B \cup C$, where $u$ is the largest label in $G$ and $|A| = k$. As we have seen, $u$ must have degree exactly $k$. If we suppose that $u$ is in the bipartite set $S_k$ of order $k$, then since $u$ must be adjacent to every vertex in the bipartite set $S_{n-k}$, it follows that $n-k \leq k$. But since by Theorem 2.1, $n-k \geq k$, therefore $k = n-k$. Thus without loss of generality we may assume that $u$ is a vertex of $S_{n-k}$, and so we may assume that $S_k = A = \{a_1, a_2, \ldots, a_k\}$, where $a_i > a_j$ whenever $i > j$, and $S_{n-k} = B \cup \{u\} = \{b_1, b_2, \ldots, b_{n-k-1}, u\}$, where $b_i > b_j$ whenever $i > j$.

From Lemma 2.2 we have $a_i + b_j \in B \cup C \cup \{u\}$ for every $i \in 1..k$, $j \in 1..n-k-1$. From Lemmas 2.2 and 2.3 it follows that $a_1 + b_j \in B \cup \{u\}$ for every $j \in 1..n-k-1$. Since $b_1 \neq a_1 + b_j$, we must have $a_1 + b_j = b_{j+1}$ for every $j \in 1..n-k-2$ and $a_1 + b_{n-k-1} = u$. But then

$$u = a_1 + b_{n-k-1} < a_2 + b_{n-k-1} < a_2 + u$$

which implies $a_2 + b_{n-k-1} = a_1 + u$.

However, since $u = a_1 + b_{n-k-1}$, it follows that $a_2 = 2a_1$, an impossibility as it would imply an edge between vertex $a_1$ and the isolate $u + a_1$. \qed

Observe that for $k = 1$, $K_2 = K_{1,1}$, while for $k = 2$, $K_3$ contains $K_{2,1}$. Thus Theorem 2.2 is sharp. On the other hand, we shall see in Section 5 that the lower bound for $n(G)$ in Theorem 2.1 is not sharp.

**Remark 2.1.** Let $d_1, d_2, \ldots, d_n$ be the degree sequence of a connected graph $G$ of order $n \geq 2$, where $d_1 \leq d_2 \leq \cdots \leq d_n$. It was shown in [4] that $\sigma(G) > \max_{1 \leq i \leq n}(d_i-i)$. As a direct consequence of this result, we have another necessary condition, namely $d_i-i \leq k-1$ for each $i = 1, 2, \ldots, n$, for $G$ to be a $k$-optimum summable graph.
3. Unit Graphs

It was pointed out in Section 1 that unit graphs and 1-optimum summable graphs are identical. Smyth [5] showed that if $G$ is a unit graph of order $n$, then $e(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$; he established further that for all integers $m$ and $n$ with $1 \leq n - 1 \leq m \leq \left\lfloor \frac{n^2}{4} \right\rfloor$, there exists a unit graph of order $n$ and size $m$. Ellingham [6] proved that any nontrivial tree is a unit graph, a conjecture of Harary [2]. Until now, however, the problem of characterizing unit graphs remains open. In this section, we describe a new family of unit graphs.

Given integers $p \geq 3$ and $q \geq 2$, let $Q(p, q)$ denote the graph obtained from the union of the cycle $C_p$ of order $p$ and the path $P_q$ of order $q$ by identifying one end-vertex of $P_q$ with a vertex of $C_p$ (see Figure 3.1). $Q(p, q)$ is called a tadpole.

![Figure 3.1. The tadpole $Q(p, q)$](image)

Our aim in this section is to show that every tadpole is a unit graph. The following observation on a generalized Fibonacci sequence will be useful.

**Lemma 3.1.** If an integer sequence $\{a_i| i = 1, 2, \cdots\}$ satisfies the following condition $(\ast)$:

\[
\begin{align*}
& a_2 > a_1 > 0 \\
& a_i = a_{i-1} + a_{i-2} \quad \text{for} \quad i \geq 3,
\end{align*}
\]

then

\[a_k + a_j < a_{j+1} \quad \text{for} \quad j - k \geq 2 \quad \text{and} \quad k \geq 1.\]

**Proof.** Since $j - k \geq 2$ and $k \geq 1$, $a_k \leq a_{j-2}$. Now $a_{j+1} - a_j = a_{j-1}$. Thus, $a_k + a_j < a_{j+1}$.

It follows from this result that if the label sequence $\{a_i| i = 1, 2, \cdots, p\}$ satisfies $(\ast)$, then $G^+(\{a_i| i = 1, 2, \cdots, p\}) \cong P_{p-1} \cup K_1$.

**Theorem 3.1.** The tadpole $Q(p, q)$ is a unit graph for all $p \geq 3$ and $q \geq 2$. 
Proof. Since $\delta(Q(p,q)) = 1$, $\sigma(Q(p,q)) \geq 1$. Let $G = Q(p,q)$, where $V(G) = A \cup B$, $A = \{u_1,u_2,\ldots,u_p\}$, $B = \{v_1,v_2,\ldots,v_{q-1}\}$, and the subgraph induced by $A$ is isomorphic to $C_p$. Let $V(G_1) = V(G) \cup \{w_1\}$. We consider two cases.

**Case 1.** $p = 3$ and $q \geq 2$.

Consider a labeling $g$ of $G_1$ as follows:

$$
\begin{cases}
g(u_1) = 1, & g(u_2) = 2, & g(u_3) = 3, & g(v_1) = 4; \\
g(v_2) = 5 & \text{for } q \geq 3; \\
g(v_i) = g(v_{i-1}) + g(v_{i-2}) & \text{for } 3 \leq i \leq q - 1; \\
g(w_1) = \begin{cases} 5 & \text{when } q = 2; \\
g(v_{q-1}) + g(v_{q-2}) & \text{when } q > 2. \\
\end{cases}
\end{cases}
$$

Let $H = G^+(\{g(x)|x \in V(G_1)\})$. We wish to prove that $H \cong G_1$.

Let $Y = \{g(v_i)|i = 1,2,\ldots,q - 1\}$. Since $Y \cup \{g(u_1)\}$ satisfies the condition $(\ast)$ in Lemma 3.1, $G^+(Y \cup \{g(u_1)\}) \cong P_{q-1} \cup \overline{K_1}$. This, together with the value of $g(w_1)$, implies that $H[Y \cup \{g(w_1)\}] \cong P_4$. Clearly, $H[\{g(u_1),g(u_2),g(u_3)\}] \cong C_3$. It is now easy to see that $H \cong G_1$, as asserted. Hence $\sigma(Q(3,q)) = 1$ for $q \geq 2$.

**Case 2.** $p \geq 4$ and $q \geq 2$.

Consider a labeling $g$ of $G_1$ as follows:

$$
\begin{cases}
g(u_1) = 1, & g(u_2) = 3; \\
g(u_i) = g(u_{i-1}) + g(u_{i-2}) & \text{for } 3 \leq i \leq p - 1; \\
g(u_p) = g(u_{p-1}) + g(u_1), & g(v_1) = g(u_p) + g(u_{p-2}); \\
g(v_2) = g(u_p) + g(u_{p-1}) & \text{for } q \geq 3; \\
g(v_i) = g(v_{i-1}) + g(v_{i-2}) & \text{for } 3 \leq i \leq q - 1; \\
g(w_1) = \begin{cases} g(u_p) + g(u_{p-1}) & \text{when } q = 2; \\
g(v_{q-1}) + g(v_{q-2}) & \text{when } q > 2. \\
\end{cases}
\end{cases}
$$

Let $J = G^+(\{g(x)|x \in V(G_1)\})$. We wish to prove that $J \cong G_1$.

The strictly increasing sequence

$g(u_1), g(u_2), \ldots, g(u_{p-2}), g(u_p), g(u_{p-1}), g(v_1), g(v_2), \ldots, g(v_{q-1}), g(w_1)$

has subsequence $X = \{g(u_1), g(u_2), \ldots, g(u_{p-2}), g(u_{p-1})\}$. Since $X$ satisfies the condition $(\ast)$ in Lemma 3.1, $G^+(X) \cong P_{p-2} \cup \overline{K_1}$. This, together with the values of $g(u_p)$, $g(v_1)$ and $g(v_2)$ (or $g(u_1)$), ensures that $J[X \cup \{g(u_p)\}] \cong C_p$.

Consider the sequence $Y = \{g(u_{p-3}), g(v_j)|j = 1,2,\ldots,q - 1\}$. Note that $Y$ satisfies the condition $(\ast)$ in Lemma 3.1, so that $G^+([g(u_{p-3})] \cup Y] \cong P_{q-1} \cup \overline{K_1}$. This, together with the value of $g(w_1)$, ensures that $J[\{g(u_{p-3})\} \cup Y] \cong P_q$.

It is clear from the definition of $g$ that $g(u_{p-3})$ is a vertex of degree 3 in $J$. Next we assert that no other adjacencies between $g(u_i)$ with $i \neq p - 3$ and $g(v_j)$ exist. Suppose
that there exist $i, j$ with $i \neq p - 3$ such that $g(u_i) + g(v_j) \in V(G_1)$. Then either $g(u_i) + g(v_j) = g(v_k)$ with $k > j$ or $g(u_i) + g(v_j) = g(w_1)$. For $q > 2$, however, $g(w_1) - g(v_j) \geq g(v_{q-2}) > g(v_p)$. Thus, $g(u_i) + g(v_j) = g(v_k)$ for some $k > j$. If $k > 2$, then $g(v_k) - g(v_j) \geq g(v_{k-2}) \geq g(v_1) > g(u_p)$, a contradiction. Thus $k \leq 2$, and we have $k = 2$ and $j = 1$. Hence $g(u_i) = g(u_{p-3})$ and so $i = p - 3$, a contradiction.

It follows from the above discussion that $J[X \cup Y \cup \{g(u_p)\}] \cong G$. Clearly, $g(w_1)$ is isolated in $J$. Hence $J \cong G_1$, as required.

This completes the proof of Theorem 3.1.

4. 2-Optimum Summable Graphs

It is known [2] that $\sigma(C_4) = 3$ and $\sigma(C_n) = 2$ for all $n \geq 3$ with $n \neq 4$. Thus $\{C_n|n \geq 3, n \neq 4\}$ is a family of 2-optimum summable graphs. In this section we introduce two new families of 2-optimum summable graphs.

Consider two tadpoles $Q = Q(p, q)$ and $Q' = Q'(p', q')$ with isolated vertices $w_1$ and $w_1'$, respectively. We first sum-label $Q \cup \{w_1\}$ and $Q' \cup \{w'_1\}$ as described in Section 3, using a labelling $g$. Observe that since under $g$ each edge is represented by a unique vertex, we can multiply the labels by any positive integer and still retain a sum labeling. Now form a single graph $B = B(p, q, p', q')$ from $Q$ and $Q'$ by adding the edge $(v_{q-1}, v'_{q'-1})$. We multiply all the original labels of $Q' \cup \{w'_1\}$ by $g(w_1)$, yielding a sum labeling $h$, and then reassign $h(w_1) \leftarrow g(w_1)g(v'_{q'-1}) + g(v_{q-1})$ to represent the new edge. Since $h(w'_1) = g(u_1)$, $B \cup \{w_1, w'_1\}$ now has a sum labeling. We have proved

**Theorem 4.1.** $B(p, q, p', q'), p, p' \geq 3, q, q' \geq 2$, is 2-optimum summable.

We now construct another 2-optimum summable graph. Given integers $p, q, r$ with $p \geq q \geq r \geq 2$ and $q \geq 3$, let $\theta(p, q, r)$ denote the graph obtained by connecting two vertices via three internally disjoint paths $P_r$, $P_q$ and $P_p$ as shown in Figure 4.1. We call the graph $\theta(p, q, r)$ a generalized $\theta$-graph.

![Figure 4.1. The generalized $\theta$-graph $\theta(p, q, r)$](image-url)
**Theorem 4.2.** The generalized $\theta$-graph $\theta(p,q,r)$ is a 2-optimum summable graph for all $p, q, r$ with $p \geq q \geq r \geq 2$ and $q \geq 3$ except when $(p,q,r) = (3,3,2)$ or when $(p,q,r) = (3,3,3)$.

**Proof.** Let $G = \theta(p,q,r)$ for $p \neq 3$ or $q \neq 3$. Let $V(G) = A \cup B$, where $A = \{u_1, u_2, \ldots, u_{q+r-2}\}$, $B = \{v_1, v_2, \ldots, v_{p-2}\}$ and the subgraphs induced by $A$ and $B$ are respectively isomorphic to $C_{q+r-2}$ and $P_{p-2}$. Since $\delta(G) = 2$, $\sigma(G) \geq 2$. Let $V(G_2) = V(G) \cup \{w_1, w_2\}$.

**Case 1.** $r = 2$, $q = 3$ and $p \geq 6$.

Consider a labeling $h$ of $G_2$ as follows:

$$
\begin{align*}
&h(u_1) = 1, h(u_2) = 2, h(u_3) = 3; \\
&h(v_1) = 4, h(v_2) = 5; \\
&h(v_i) = h(v_{i-1}) + h(v_{i-2}) \quad \text{for} \quad 3 \leq i \leq p - 2; \\
&h(w_1) = h(v_{p-2}) + h(u_2); \\
&h(w_2) = h(v_{p-2}) + h(v_{p-3}).
\end{align*}
$$

Let $H = G^+ (\{h(x) | x \in V(G_2)\})$.

Clearly, $u_1$ and $u_2$ are two vertices of degree 3 in $H$. Since $p \geq 6$, $h(v_{p-3}) - h(u_2) > h(v_{p-4})$. Thus, $h(w_1)$ is isolated in $H$. By means of an argument similar to that given in Case 1 of the proof of Theorem 3.1, it is not difficult to verify that $H \cong G_2$. The result thus follows.

**Case 2.** $q + r \geq 6$ and $p \geq 6$.

Consider a labeling $h$ of $G_2$ as follows:

$$
\begin{align*}
&h(u_1) = 1, h(u_2) = 3; \\
&h(u_i) = h(u_{i-1}) + h(u_{i-2}) \quad \text{for} \quad 3 \leq i \leq q + r - 3; \\
&h(u_{q+r-2}) = h(u_{q+r-3}) + h(u_1); \\
&h(v_1) = h(u_{q+r-2}) + h(u_{q+r-4}), h(v_2) = h(u_{q+r-2}) + h(u_{q+r-3}); \\
&h(v_i) = h(v_{i-1}) + h(v_{i-2}) \quad \text{for} \quad 3 \leq i \leq p - 2; \\
&h(w_1) = \begin{cases} 
    h(v_{p-2}) + h(u_{q-2}) & \text{when} \quad r = 2, \\
    h(v_{p-2}) + h(u_{q+1}) & \text{when} \quad r = 3, 4 \\
    h(v_{p-2}) + h(u_{q-4}) & \text{when} \quad r \geq 5
\end{cases} \\
&h(w_2) = h(v_{p-2}) + h(v_{p-3}).
\end{align*}
$$

Let $J = G^+ (\{h(x) | x \in V(G_2)\})$.

Clearly, the degree of $u_{q+r-5}$ is 3 and $u_1 u_2 \cdots u_{q+r-5} u_{q+r-4} u_{q+r-2} u_{q+r-3} u_1$ is a cycle of order $q + r - 2$ in $J$. Since $p \geq 6$,

$$
h(v_{p-3}) - \max\{h(u_{q-2}), h(u_{q+1}), h(u_{q-4})\} > h(v_{p-4}).$$
Thus, \( h(w_1) \) is isolated in \( J \). By means of an argument similar to that given in Case 2 of the proof of Theorem 3.1, it is not difficult to verify that \( J \cong G_2 \). The result thus follows.

**Case 3.** \( p \leq 5 \).

The following labeling-induced sum graphs show that this case is also covered.

\[
G^+(\{1, 3, 4, 7, 11, 18, 29, 30, 48, 59, 107, 108, 166\}) \cong \theta(5, 5, 5) \cup K_2
\]
\[
G^+(\{1, 3, 4, 7, 11, 18, 19, 30, 37, 67, 68, 104\}) \cong \theta(5, 5, 4) \cup K_2
\]
\[
G^+(\{1, 3, 4, 7, 11, 12, 19, 23, 42, 43, 65\}) \cong \theta(5, 5, 3) \cup K_2
\]
\[
G^+(\{1, 3, 4, 7, 8, 12, 15, 27, 31, 42\}) \cong \theta(5, 5, 2) \cup K_2
\]
\[
G^+(\{1, 3, 4, 7, 11, 18, 19, 30, 37, 38, 67\}) \cong \theta(5, 4, 4) \cup K_2
\]
\[
G^+(\{1, 3, 4, 7, 11, 18, 19, 30, 31, 37\}) \cong \theta(5, 4, 3) \cup K_2
\]
\[
G^+(\{1, 3, 4, 5, 8, 9, 17, 20, 26\}) \cong \theta(5, 4, 2) \cup K_2
\]
\[
G^+(\{1, 3, 4, 7, 11, 12, 19; 23, 31\}) \cong \theta(5, 3, 3) \cup K_2
\]
\[
G^+(\{1, 3, 4, 7, 8, 12; 13, 15\}) \cong \theta(5, 3, 2) \cup K_2
\]
\[
G^+(\{1, 3, 4, 7, 11, 12, 19, 23, 34, 42\}) \cong \theta(4, 4, 4) \cup K_2
\]
\[
G^+(\{1, 3, 4, 7, 8, 12, 15; 22, 27\}) \cong \theta(4, 4, 3) \cup K_2
\]
\[
G^+(\{7, 8, 11, 15, 19, 23, 30, 34\}) \cong \theta(4, 4, 2) \cup K_2
\]
\[
G^+(\{1, 3, 4, 7, 8, 12; 15, 20\}) \cong \theta(4, 3, 3) \cup K_2
\]
\[
G^+(\{1, 3, 4, 7, 8; 9, 11\}) \cong \theta(4, 3, 2) \cup K_2
\]

This completes the proof of Theorem 4.2. \( \square \)

**Remark 4.1.** The two generalized \( \theta \)-graphs not included in Theorem 4.2 are \( \theta(3, 3, 2) \) and \( \theta(3, 3, 3) \). They are, as a matter of fact, not 2-optimum summable graphs. Indeed, by Theorem 2.2, we have \( \sigma(\theta(3, 3, 2)) \geq 3 \) and \( \sigma(\theta(3, 3, 3)) \geq 3 \). These, together with the two labeling-induced sum graphs

\[
G^+(\{2, 4, 7, 9; 6, 11, 16\}) \cong \theta(3, 3, 2) \cup K_3,
\]
\[
G^+(\{1, 2, 3, 8; 10; 4, 11, 18\}) \cong \theta(3, 3, 3) \cup K_3.
\]

show that \( \sigma(\theta(3, 3, 2)) = \sigma(\theta(3, 3, 3)) = 3 \).

5. \( k \)-Optimum Summable Graphs, \( k \geq 3 \)

In this final section we shall establish two existence results, one for 3-optimum summable graphs and one for \( k \)-optimum summable graphs, where \( k \geq 4 \).

**Theorem 5.1.** For each \( l \geq 1 \), there exists a 3-optimum summable graph of order \( 4l + 3 \).
Proof. Given \( l \geq 1 \), our aim is to construct a subset \( S^l \) of \( N \) such that \( G^+(S^l) \cong G_3 \) and to show that \( G \) is a \( 3 \)-optimum summable graph of order \( 4l + 3 \).

Let \( A_i = \{a_{i1}, a_{i2}, a_{i3}\} \) for \( 1 \leq i \leq l + 2 \) and \( B = \{b_1, b_2, \ldots, b_l\} \), where
\[
\begin{align*}
a_{11} &= 1, \quad a_{12} = 4 \quad \text{and} \quad a_{13} = 7; \\
a_{ij} &= \sum_{p=1}^{3} a_{(i-1)p} - a_{(i-1)j} \quad \text{for} \; 2 \leq i \leq l + 2 \quad \text{and} \quad 1 \leq j \leq 3; \\
b_i &= \sum_{p=1}^{3} a_{ip} \quad \text{for} \; 1 \leq i \leq l.
\end{align*}
\]

Let \( S^l = (\cup_{i=1}^{l+2} A_i) \cup B \) and \( H = G^+(S^l) \). Clearly, \( v(H) = 4l + 6 \).

For \( i \geq 3 \) and \( 1 \leq j \leq 3 \), observe that
\[
a_{ij} = \sum_{p=1}^{3} a_{(i-1)p} - a_{(i-1)j} \\
= 2 \sum_{p=1}^{3} a_{(i-2)p} - \left( \sum_{p=1}^{3} a_{(i-2)p} - a_{(i-2)j} \right) \\
= \sum_{p=1}^{3} a_{(i-2)p} + a_{(i-2)j} > a_{(i-1)j}. \tag{\#}
\]

Clearly, \( \min\{a_{(l+2)1}, a_{(l+2)2}, a_{(l+2)3}\} > b_i \) for \( 1 \leq i \leq l \). Thus, the three vertices in \( A_{l+2} \) are the three largest vertices in \( H \). For \( 1 \leq j_1 \leq 3, 1 \leq j_2 \leq 3 \) and \( j_1 \neq j_2 \),
\[
a_{(l+2)j_1} - a_{(l+2)j_2} = a_{(l+1)j_2} - a_{(l+1)j_1} = \cdots = (-1)^{l+1}(a_{1j_1} - a_{1j_2}).
\]

Now \( A_2 = \{5, 8, 11\} \) and \( a_{1j_1} - a_{1j_2} \) can only take one of the two positive integers 3 and 6. Thus \( a_{1j_1} - a_{1j_2} \notin S^l \), and so the three vertices in \( A_{l+2} \) are isolated in \( H \).

It follows from the values of the three integers in \( A_{i+1} \) that \( H[A_i] \cong C_3 \) for \( 1 \leq i \leq l + 1 \). Notice that
\[
a_{ij} + a_{(i-1)j} = \sum_{p=1}^{3} a_{(i-1)p} - a_{(i-1)j} + a_{(i-1)j} = \sum_{p=1}^{3} a_{(i-1)p} = b_{i-1}
\]
for \( 1 \leq i \leq l + 1 \). This implies that \( a_{ij} \) is adjacent to \( a_{(i-1)j} \). Thus the degree of any vertex in \( \cup_{i=1}^{l+1} A_i \) is at least 3.

For \( 1 \leq i \leq l \) and \( 1 \leq j \leq 3 \), by (\#), we have
\[
a_{(i+2)j} = \sum_{p=1}^{3} a_{ip} + a_{ij} = b_i + a_{ij}.
\]

Thus \( b_i \) is adjacent to \( a_{i1}, a_{i2}, a_{i3} \) for \( 1 \leq i \leq l \), and so the degree of any vertex in \( B \) is at least 3.
Let $G = H[S^l \setminus A_{l+2}]$. It follows from the above discussion that $G$ is connected and $\delta(G) = 3$. Thus $G$ is a 3-optimum summable graph of order $4l + 3$. The proof is thus complete.

As an illustration of the construction used in the above proof, we present the graph $G^+(S^2)$ in Figure 5.1.

Finally, we have:

**Theorem 5.2.** For each $k \geq 4$, there exists a $k$-optimum summable graph.

**Proof.**

Given $k \geq 4$, our aim is to construct a subset $S^{(k)}$ of $N$ such that $G^+(S^{(k)}) \cong G_k$ and to show that $G$ is a $k$-optimum summable graph.

Let $I = \{1, 2, \ldots, k\}$ and $a_i = 10^{i-1}$ for $i \in I$. Define

$$
\begin{align*}
A_j &= \{ \sum_{p \in D} a_p | \ D \subseteq I \ \text{and} \ |D| = j \} \ \text{for} \ 1 \leq j \leq k; \\
B &= \{ a_i + \sum_{p=1}^{k} a_p | i \in I \}.
\end{align*}
$$

Let $S^{(k)} = \bigcup_{j=1}^{k} A_j \cup B$ and $H = G^+(S^{(k)})$.

Clearly, the $k$ vertices of $B$ are the $k$ largest vertices in $H$. Since $u - v \notin S^{(k)}$ for any pair of distinct vertices $u, v \in B$, the $k$ vertices in $B$ are isolated in $H$.

It is obvious that $|A_k| = 1$ and the vertex in $A_k$ is adjacent to all the $k$ vertices of $A_1$. For any vertex $w \in A_j$, where $1 \leq j < k$, there exists a subset $D$ of $I$ with $|D| = j$ such that $w = \sum_{p \in D} a_p$. Clearly, $w$ is adjacent to $a_p$ for $p \in I \setminus D$. For a fixed $\alpha \in D$, by the fact that $w + (\sum_{p \in I \setminus D} a_p + a_\alpha) = \sum_{p \in I} a_p + a_\alpha$, $w$ is adjacent to $\sum_{p \in I \setminus D} a_p + a_\alpha$ which is a vertex of $A_{k-j+1}$. Thus, $d(w) \geq |I \setminus D| + |D| = k$. Let $G = H[S^{(k)} \setminus A_k]$.

It follows from the above discussion that $G$ is connected and $\delta(G) = k$. Hence $G$ is a $k$-optimum summable graph.

Figure 5.1
References


