Correction to
AN OPTIMAL ALGORITHM TO COMPUTE ALL THE COVERS OF A STRING

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ABSTRACT

This note corrects an error in a paper recently published in this journal (An optimal algorithm to compute all the covers of a string, IPL 50-5 (1994) 239-246). The correction consists primarily of a new subalgorithm which is called by COMPUTE_COVERS, the main algorithm presented in the paper referenced. It turns out that the new subalgorithm is itself sufficient to solve the original problem — that is, to compute all the covers of a given string in time linear in the string length — and so it is presented here as a self-contained algorithm in its own right.

1 INTRODUCTION

For notation and terminology see [1]. The error in [1] relates to case (c) of Theorem 2.1, where it is shown that the proper covers of a given string \( x = v^*vv^* \) must in fact be covers of \( v^* \) which also cover \( x \). (Here \( v^*v = x[1..k_1] \) and \( v^* = x[1..k_2] \) are substrings identified when \( x \) is expressed in normal form.) Thus the problem of computing the proper covers of \( x \) is reduced to the problem of computing the covers \( u \) of \( v^* \), provided that it can be efficiently checked that these covers \( u \) are also covers of \( x \). On page 244 of [1] the following statement is made:

Recall that a cover \( u \) of \( v^* \) must be both a prefix and a suffix of \( v^* \). Thus \( u \) is a cover of \( v^*v^* \) if and only if, in the substring \( vv^* = x[k_2+1,k_1+k_2] \), there exist at most \(|u| - 1 \) consecutive positions \( i \) such that \( f[i] < |u| \).

(Here \( f[i] \) is an element of the “failure array” \( f = f[1..n] \) and specifies the length of the longest border of \( f[1..i] \).) As pointed out in [2], this statement is incorrect. In order to formulate a correct “if and only if” condition, we first introduce a definition: given nonnegative integers \( i \) and \( h \), we say that \( i \) reduces to \( h \) iff there exists a positive integer \( j \) such that \( f^j[i] = h \) (where \( f^j \) denotes \( j \) compositions of \( f \)). The following result then leads to a correct statement of the condition:

Lemma 1.1 Let \( h \) and \( r \) denote integers satisfying \( 1 \leq h \leq r \leq n \), and suppose that \( u = x[1..h] \) is a cover of \( x[1..r] \). Then \( u \) is a cover of \( x \) if and
only if there exist at most $h - 1$ consecutive positions of $x[r + 1..n]$ which do not reduce to $h$.

Proof If $u$ is a cover of $x$, then the substring $x[1..h]$ begins in at least every $h^{\text{th}}$ position of $x$, and every such occurrence terminates in the letter $x[h]$ at some position $i$, where for $i \geq h$, it follows that $i$ reduces to $h$.

Now suppose that at least every $h^{\text{th}}$ position of $x[r + 1..n]$ reduces to $h$. Observe that since $u$ is a cover of $x[1..r]$, it must be true that at least every $h^{\text{th}}$ position of $x[h + 1..x]$ reduces to $h$. Thus at least every $h^{\text{th}}$ position of $x[h + 1..n]$ reduces to $h$, and each of these positions marks the end of an occurrence of $u$. Hence $u$ covers $x$. \Halmos

In view of Lemma 1.1, a correct necessary and sufficient condition for a cover $u$ of $v^*$ to also be a cover of $x = v^* vv^*$ is that there should exist at most $|u| - 1$ consecutive positions $i$ of $vv^*$ such that $i$ does not reduce to $|u|$. Consider then an integer $k'$ which is constrained to be the length of a border of $vv^*$. It turns out that the stated condition can be efficiently implemented by performing (at most once) a subalgorithm which computes the greatest of these integers $k'$ such that at most $k' - 1$ consecutive positions of $vv^*$ do not reduce to $k'$. Then having found such a greatest value of $k'$, it suffices, in order to check that a cover $u$ of $v^*$ is also a cover of $x$, to compare $|u|$ with $k'$, a constant time operation: $u$ will be a cover of $x$ if and only if $|u| \leq k'$.

It turns out further that the subalgorithm which computes the greatest value of $k'$ is merely a special case of an algorithm which considers in turn the borders of $x$ itself, deciding for each one whether or not it is in fact a cover of $x$. It is this slightly more general algorithm that is described in the next section.

## 2 Deciding Whether a Border of $x$ is a Cover

We suppose from now on that $x$ is an arbitrary string of length $n$ with exactly $m$ borders $x[1..b_1], x[1..b_2], \ldots, x[1..b_m]$, where for $j = 1, 2, \ldots, m$, $b_j = f^j[n]$, while $f^{m+1}[n] = f[b_m] = 0$. Note that $b_1 > b_2 > \cdots > b_m$; to exclude trivial cases, we assume without loss of generality that $b_m > i_0$, where $i_0$ is the greatest integer for which $f[i_0] = 0$. Our task is to determine whether or not each $x[1..b_j]$ is a cover of $x$. Applying Lemma 1.1 with $h = r = b_j$, we see that this task is equivalent to determining whether or not it is true that there exist at most $b_j - 1$ consecutive positions of $x[b_0 + 1..n]$ which do not reduce to $b_j$.

The main idea used in our algorithm is that of a “border tree”, which we now define. A border tree $B_x$ is a rooted tree in which each node has a unique integer label chosen from $[0..n]$: the root has label 0, and the parent of the node with label $i$, $i = 1, 2, \ldots, n$, is the node with label $f[i]$. It is clear that $B_x$ is in fact a tree, and we observe that the descendants of the node labelled $i$ in $B_x$ are exactly those nodes whose labels reduce to $i$. Thus by arranging these descendant nodes in ascending label sequence, we can easily determine whether any difference between adjacent labels in the sequence exceeds $i$ or not; if not, and if $x[1..i]$ is a border of $x$, then by Lemma 1.1 we are entitled to conclude that $x[1..i]$ is also a cover of $x$. 


The algorithm first considers the shortest border $x[1..b_m]$; a collection of all the nodes in the subtree of $B_x$ rooted at $b_m$ is formed, and these nodes are sorted, using a binsort, into ascending label sequence. These sorted labels are then added into a doubly-linked list $L_m$ whose initial element is a dummy element with label 0; as each label is added to the list, the difference between the current and the preceding label is computed, so that the quantity $\text{MAX}\_\text{GAP}$, the maximum difference between adjacent labels, can be maintained. When $L_m$ has been fully updated, $x[1..b_m]$ will be a cover if and only if $b_m \geq \text{MAX}\_\text{GAP}$.

The algorithm now considers each $b_j$ in turn, $j = m - 1, m - 2, \ldots, 1$. For each $j$, the labels contained in the subtree rooted at $b_{j+1}$ but NOT in the subtree rooted at $b_j$ are deleted from $L_{j+1}$, yielding $L_j$. (Observe that $b_{j+1}$ is necessarily the parent of $b_j$ in the border tree because $b_{j+1} = f[b_j]$.) As each label is deleted, the preceding and following labels are inspected, and the difference between these labels is computed; if this difference exceeds $\text{MAX}\_\text{GAP}$, then $\text{MAX}\_\text{GAP}$ is updated with the computed difference. When the formation of $L_j$ is complete, $x[1..b_j]$ will again be a cover if and only if $b_j \geq \text{MAX}\_\text{GAP}$.

In order to be able to handle the deletions from $L_{j+1}$ corresponding to $b_j$ in time proportional to the number of nodes deleted, a strategy needs to be implemented which identifies those nodes. One way to accomplish this is to introduce at each position in the subtree rooted at $b_m$ a pointer to the corresponding position in $L_m$; then the deletions can be effected by traversing the appropriate subtrees of $b_{j+1}$.

The algorithm is summarized below, with notes indicating the time required for each step.

1. Compute the failure array $f = f[1..n]$ in time $\Theta(n)$.
2. Compute the border tree $B_x$ in time $\Theta(n)$. In practice $B_x$ will be implemented as an equivalent binary tree. Also the construction of $B_x$ will usually be limited only to those nodes occurring in the subtree rooted at $b_m$.
3. Perform the binsort of the labels in the subtree of $B_x$ rooted at $b_m$. This requires $\Theta(n)$ time. Then, in $O(n)$ time, construct the doubly-linked list $L_m$ and compute $\text{MAX}\_\text{GAP}$. If $b_m \geq \text{MAX}\_\text{GAP}$, then output $b_m$.
4. For each $j = m - 1, m - 2, \ldots, 1$, compute $L_j$ from $L_{j+1}$ in time proportional to the number of nodes deleted. At the same time, recompute $\text{MAX}\_\text{GAP}$; if $b_j \geq \text{MAX}\_\text{GAP}$, then output $b_j$. The time required over all values of $j$ is proportional to the number of nodes in $L_m$, hence $O(n)$.

This algorithm computes all the covers of $x = x[1..n]$ in $\Theta(n)$ time and $\Theta(n)$ space. Observe that, with trivial modifications, the same algorithm can be used to compute all the covers of any prefix $x[1..i]$ in $\Theta(i)$ time and $\Theta(i)$ space.

As an example of the algorithm, consider the string

$$x = (ab)^6(aab)^3aba$$

with failure array

$$f = (0, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 1, 2, 3, 1, 2, 3, 4, 5),$$
which yields the border tree shown in Figure 2.1. Since $f[21] = 5$ and $f[5] = 3$, we find $m = 3$ and $b_3 = 3$, $b_2 = 5$, $b_1 = 21$. The initial list $L_3 = 0 - 3 - 5 - 7 - 9 - 11 - 13 - 16 - 19 - 21$, from which we compute $\text{MAXGap} = 3$ and so accept $x[1..b_3] = aba$ as a cover. For $j = 2$ we compute $L_2 = 0 - 5 - 7 - 9 - 11 - 13 - 21$ and $\text{MAXGap} = 8$, and thus reject $x[1..b_2] = ababa$ as a cover. Finally, we compute $L_1 = 0 - 21$ and accept $x = x[1..b_1]$ as a cover.

REFERENCES


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