Disturbances to the uniform stream flow of a fluid with a free surface

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This thesis is presented for the degree of Doctor of Philosophy of Murdoch University
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I declare that this thesis is my own account of my research except where other sources are fully acknowledged. It contains as its main content work which has not previously been submitted for a degree at any tertiary education institution.

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Abstract

We consider two main problems involving disturbances to uniform stream flow. In the first part of the thesis we examine flow over topography with particular interest in obtaining subcritical solutions with no downstream waves. Solutions to the linearised, fully nonlinear and weakly nonlinear problem are computed for two different types of topography. An array of waveless solutions corresponding to one or more trapped waves are computed at a range of different Froude numbers and are shown to provide a rather elaborate mosaic of solution curves in parameter space. The free surface is shown to evolve into different shapes as we track these waveless contours through parameter space. In addition, for one type of bottom topography, certain values of the dimensionless flow rate and obstruction height are shown to have waveless solutions for almost all obstruction separation distances greater than some particular value. In the second part of the thesis we examine the flow past a line sink. We consider the fully nonlinear problem with the inclusion of surface tension and investigate the maximum sink strength for a given stream flow, before examining non-unique solutions. The addition of surface tension allows for a more thorough investigation into the characteristics of the solutions and produces some interesting results. The breakdown of steady solutions with surface tension appears to be caused by a curvature singularity as the flow rate approaches the maximum. The non-uniqueness in solutions is shown to occur for a range of parameter values in all cases with non-zero surface tension. The work involved in this thesis has application in design of submerged structures and water quality management in reservoirs.
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# Contents

1 Introduction .......................................................... 1

2 Literature Review ....................................................... 4
   2.1 Finite depth flows over topography .......................... 4
   2.2 Flows due to a source or sink ................................. 7

3 Finite depth flows. Part I; Linearised methods for flow past arbitrary topography ................................................. 12
   3.1 Problem formulation ............................................. 12
   3.2 Linearised method ................................................ 14
       3.2.1 Semi-elliptical obstructions .......................... 16
       3.2.2 Gaussian obstructions ................................. 19

4 Finite depth flows. Part II; Nonlinear problem formulation and numerical methods .................................................. 21

5 Finite depth flows. Part III; Nonlinear problem of flow past a semi-ellipse .............................................................. 25
   5.1 Results .............................................................. 26
   5.2 Summary ............................................................ 34
6 Finite depth flows. Part IV; Nonlinear problem of flow past Gaussian topography 35
   6.1 Results .................................. 36
      6.1.1 Contour behaviour .................... 36
      6.1.2 Solution behaviour; obstructions of positive height 45
      6.1.3 Solution behaviour; obstructions of negative height 48
   6.2 Summary .................................. 60

7 Finite depth flows. Part V; Weakly nonlinear analysis 61
   7.1 Weakly nonlinear problem ................... 62
   7.2 Results .................................. 63

8 Infinite depth flows. Nonlinear problem of a line sink in a flowing stream 72
   8.1 Problem formulation ......................... 73
   8.2 Linearised problem .......................... 75
   8.3 Nonlinear problem and numerical method ....... 76
   8.4 Results .................................. 80
      8.4.1 General behaviour ....................... 80
      8.4.2 Non-unique solutions .................... 90
   8.5 Summary .................................. 92

9 Conclusion 93
   9.1 Finite depth flows over topography ............ 93
   9.2 Infinite depth flow past a line sink ............ 94
   9.3 Concluding remarks .......................... 95
Chapter 1

Introduction

Problems involving disturbances to a fluid with a free surface have a long history in fluid dynamics [1, 47, 55, 68], and have a wide range of important applications. Some disturbances that have been studied include those resulting from topography at the bottom of the fluid, fully or partially submerged objects and the withdrawal of fluid through a sink. Applications include atmospheric flows in the lee of mountains [3, 22, 23], water quality management [33, 36, 37, 38, 42], ship flows and flow past underwater structures such as oil platforms, bridges or weirs [13, 65, 67].

We consider two main problems involving disturbances to uniform stream flow. The first problem we will consider is the classic problem of steady flow of an ideal fluid in a stream of finite depth that is disturbed by an obstruction on the stream bed. The fluid is flowing at a uniform speed upstream, and as the fluid hits the obstruction, the free surface of the fluid is disturbed. In particular, our interest in this work is in those solutions for which the surface returns to its original level with no downstream waves.

Waves that are created as a result of the flow past a submerged object
are a source of significant drag, and can make the object visible on the
surface if the waves are of significant magnitude [17, 18]. The existence
of particular disturbances that do not generate waves is of great interest
in engineering and scientific applications as it can reduce the costs of
building structures such as undersea pipelines and oil platforms, and in
defence applications, making submarines invisible to surface monitoring.
Such solutions are rare and depend on the flow conditions and shape of
the disturbance. Understanding such flows in two dimensions will help in
determining possible extensions into the much more complicated three-
dimensional case, see for example [58].

We investigate this problem using two different types of bottom topo-
graphy. We consider the case of a semi-ellipse on the stream bed, in
which waves generated by the front of the ellipse must be cancelled by
waves generated by the back of the ellipse in order for waveless solutions
to occur. We also consider the case of two Gaussian shaped obstructions
on the stream bed, in which waveless solutions occur when waves from
the second obstruction cancel the waves from the first obstruction. We
start by defining the problem with arbitrary topography on the stream
bed and examining a linearised problem, with a brief look at the be-
haviour of the solutions with the different bottom topography. We then
investigate the fully nonlinear problem, and examine in detail the re-
sults for semi-elliptical and Gaussian shaped obstructions. In each case,
contours in parameter space representing waveless solutions are found.
Following this, we use a weakly nonlinear analysis in the form of the
Korteweg-de Vries equation to consider the problem of flow over Gauss-
ian obstructions. Comparisons are made with the results of the fully
nonlinear problem in order to examine the suitability of this method.
The work involved in the fully nonlinear problem with most of the results for the flow over semi-elliptical and Gaussian shaped obstructions has been published in the papers *A note on waveless subcritical flow past a submerged semi-ellipse* [39] and *Waveless subcritical flow past symmetric bottom topography* [41].

The second major problem we consider is the steady, uniform flow of an ideal fluid of infinite depth which is disturbed by a line sink of constant strength located at a fixed depth. We examine the general behaviour of the solutions and the effects of the added upstream flow as well as the effects of surface tension. We then investigate the maximum upstream flow for given sink strength which leads to the question of possible reasons for solution breakdown. Following this, we find non-unique solutions by forcing an increase in the deflection of the free surface.
Chapter 2

Literature Review

2.1 Finite depth flows over topography

Solutions to the linearised problem of flow past arbitrary topography on the stream bed were developed by Lamb [46], with related problems examined by Havelock [29, 30]. Since the work of Lamb [46], this general problem has been studied extensively, with summaries of different aspects of the problem given in the review articles by Long [47] and Baines [1].

Solutions to the full nonlinear problem were obtained numerically by Forbes [17, 18], who considered a semi-elliptical obstruction on the stream bed, and Forbes and Schwartz [26] and Vanden-Broeck [62], who computed nonlinear solutions for flow over a semi-circular obstruction. Forbes [20] also computed hydraulic fall flows over a semi-circular obstruction, which were verified by comparison with the results of a laboratory experiment. Belward [2] computed fully nonlinear solutions for flow over two obstructions but only considered the case with critical flow over the first obstruction and supercritical flow over the second.

King and Bloor [44] used conformal mapping to calculate the flow
over arbitrary topography, with Gaussian, triangular and semicircular obstructions examined. Higgins et al. [31] obtained solutions for symmetric and asymmetric mountain shapes on the stream bed by using a series-solution method. Zhang and Zhu [70, 71] computed the flow over a semicircular obstruction using a hodograph method and also computed the second order perturbation solution for flow over a semicircular trench which they then compared with solutions to the full nonlinear problem.

Weakly nonlinear solutions have been found using long wave theory and the Korteweg-de Vries (KdV) equation. Pratt [52] conducted an experiment which considered steady flow over two obstructions. He compared the results with long wave theory and found that in some circumstances the theory would be inconsistent with the experimental results. Dias and Vanden-Broeck [14, 16] used KdV theory to find critical solutions over a single obstruction, but these had waves upstream. They found that if they introduced a second obstruction, the waves would be trapped between the two obstructions. Binder et al. [5, 6] computed the flow over two triangular obstructions and then a single rectangular obstruction on the stream bed, using both nonlinear methods and KdV theory. They commented that for subcritical flow, the downstream waves could be eliminated by adjusting the obstruction separation in the case of the two triangles, or the length in the case of the rectangular obstruction. In both cases, however, there was no further investigation into the parameter values that result in these solutions. Choi [11] used the KdV equation with a negative forcing term to calculate flows over a single semicircular trench.

Lustri et al. [48] considered the flow past submerged bumps and trenches with inclined sides in the asymptotic limit of small Froude num-
ber. A number of solutions were computed with trapped waves above the bump or trench and no waves downstream. An inverse method was used by Binder et al. [4] to find the stream bed topography for a given waveless free surface, with multiple cases examined.

Some interesting related free surface flow problems have been studied, for example Crapper [9] and Schwartz [54] examined the flow due to an applied overpressure, such as that caused by hovercraft, and Blyth and Vanden-Broeck [7] computed the flow over a trapped bubble. Tuck [57] and Scullen and Tuck [56] considered the flow past cylinders submerged in a fluid of finite depth and Tuck [58], Tuck and Tulin [60] and Tuck and Scullen [59] each examined flows past submerged objects with no downstream waves and hence zero wave drag on the objects. These problems are of interest, but we limit our area of focus to those papers closer to our current research.

Of particular interest is the work of Forbes [18], who found nonlinear solutions for subcritical flow over a semi-elliptical obstruction on the stream bed with no downstream waves. Forbes [18] found that given a fixed Froude number $F$ and ellipse height $\beta$, the ellipse half lengths $\alpha$ could be found for which waveless solutions exist. Forbes [18] computed the first four values of $\alpha$ for different ellipse heights given $F = 0.5$ and plotted contours showing the ellipse dimensions for which waveless solutions exist. He found that the first and second nonlinear contours merged together at the limiting value of the ellipse height.

These waveless subcritical solutions are those of interest and will be the main focus of Chapters 3, 5, 6 and 7. In Chapter 3 we formulate the problem and consider a linearised method for flow over arbitrary topography. We extend the work of Forbes [18] in Chapter 5 by computing
nonlinear solutions for flow over a semi-ellipse on the stream bed and investigate the parameter values that result in waveless solutions. We then further extend the work of Forbes [18] in Chapter 6 by computing nonlinear solutions for flow over two obstructions and also flow over two trenches, both Gaussian in shape. Contours are found that show the obstruction dimensions at which solutions have no waves downstream of the second obstruction. We briefly investigate the suitability of the KdV equation in Chapter 7 by computing waveless solutions for flow over two Gaussian obstructions and comparing with the fully nonlinear solutions.

2.2 Flows due to a source or sink

The flow due to a submerged source or sink has been the subject of extensive research. The cases of infinite or finite depth fluid, two dimensional or three dimensional and steady or unsteady flows have been considered in single or multi-layered fluids. Of particular interest is the steady two dimensional flow due to a line source or sink in a single layer of fluid, for which two main solution types exist.

In the case of the infinite depth problem, the only parameter is the nondimensional flow rate, or Froude number, relative to the strength and depth of the source or sink and defined as $F_S = m/\sqrt{gH_S^3}$, where $m$ is the strength of the source/sink, $g$ is the downward acceleration due to gravity and $H_S$ is the depth of the source/sink. In the equivalent finite depth problem, the Froude number can be defined in terms of the source/sink depth, $F_S$ as before, or in terms of the fluid depth, $F_B = m/\sqrt{gH_B^3}$, where $H_B$ is the depth of the fluid. We also have the extra parameter of the relative sink depth, $H_S/H_B$. 
One of the two main steady solution types exists for small Froude number, in which a stagnation point is present on the free surface directly above the source or sink. The other main type of steady solution occurs at a higher Froude number and involves a cusp in the free surface directly above the source or sink. Many researchers (see for example [49, 50, 63, 64] and references therein) have focused on the possible existence of waves on the free surface, with discussion continuing.

Solutions with a cusp in the surface of the fluid were first found by Sautreaux [53], who used an inverse method to obtain solutions to the problem of a sink located on a boundary. The boundary was vertical above the sink and at an angle of 30 degrees to the horizontal below the sink. This problem was also examined by Craya [10], who considered various angles of the lower boundary, and experimental results for the same problem were obtained by Gariel [27]. Further experimental work with the sink located on the bottom of a channel was conducted by Harleman and Elder [28], Wood and Lai [69], Jirka [43] and Hocking [32]. Almost all of the experimental work resulted in drawdown occurring at a Froude number lower than predicted by the analytical and numerical work.

The earliest work with a stagnation point on the free surface is thought to be that of Peregrine [51], who examined the two dimensional steady flow due to a line source in a fluid of infinite depth. Approximate solutions were found using a series expansion. Vanden-Broeck et al. [66] also used a series expansion to obtain stagnation point solutions for small Froude number. This work was extended by Tuck and Vanden-Broeck [61] who used a series substitution method to find solutions with a cusp in the free surface above the source or sink at an isolated value of
the Froude number. They also mentioned stagnation point solutions, and speculated that these solutions were confined to the region $0 < F_S < 2$.

The stagnation point solutions of Tuck and Vanden-Broeck [61] were reconsidered by Hocking and Forbes [35], whose investigation revealed that these stagnation point solutions exist for Froude numbers in the region $0 < F_S < 1.42$. Mekias and Vanden-Broeck [50] found stagnation point solutions with waves in the far field for the flow due to a line source in a fluid of finite depth. These solutions only existed for Froude numbers smaller than some critical value (for each value of the source depth) and the wave amplitude was found to be exponentially small as the Froude number approached zero. This problem was also considered by Hocking and Forbes [36], who defined the regions in parameter space in which stagnation point solutions as well as cusp solutions were found to exist. The stagnation point solutions were confined to a region below $F_B = 0.25$, while the cusp solutions were confined to a region above $F_B = 1$. The maximum $F_B$ for the stagnation point region corresponded to the sink being located on the bottom of the fluid, $H_S/H_B = 1$. They also searched for steady solutions with waves in the interval between these two regions, but no results with waves were obtained.

Solutions with a stagnation point on the free surface were further investigated by Forbes and Hocking [24] for the problem of a line sink in a fluid of infinite depth with the inclusion of surface tension. Surface tension was found to have a significant effect on the solutions, with a large increase in the maximum Froude number at which stagnation point solutions exist when surface tension was included. A non-uniqueness in the solutions was found by increasing the deflection of the free surface.

Vanden-Broeck [64] used a series truncation procedure to examine
the flow due to a line source submerged in a fluid of infinite depth. A train of waves of constant amplitude was found in the far field, but it was determined that these could not exist in the full nonlinear problem. It was suggested that these solutions however, could be used to approximate nonlinear solutions, as the viscosity in a real fluid would dampen the train of waves, or, if the Froude number is small enough, the waves are so small they can be neglected.

Hocking and Forbes [37] used a boundary integral equation method to find stagnation point solutions to the problem of flow due to a line sink in a fluid of finite depth. Steady waves were found when small surface tension values were used, but were not present for large surface tension values. A non-uniqueness in solutions was found with two solutions obtained for certain parameter values.

Lustri et al. [49] examined stagnation point solutions to the problem of steady flow due to a line source in fluid of finite depth using exponential asymptotics. They showed that waves of exponentially small amplitude exist in the limit as the Froude number tends to zero and that these results are contrary to those from algebraic power series expansions that predict a flat free surface in the far field. They concluded that their approach showed that there are no steady subcritical solutions for flow due to a line sink in a fluid of finite depth with a stagnation point on the free surface directly above the sink.
The problem we will consider is that of the two-dimensional, steady flow due to a line sink in a flowing stream of infinite depth, to be examined in Chapter 8. We aim to reproduce the results of Forbes and Hocking [24], before including the stream. It is of particular interest to investigate the possibility of non-unique solutions when an added upstream flow is present.
Chapter 3

Finite depth flows. Part I; Linearised methods for flow past arbitrary topography

3.1 Problem formulation

We will consider the problem of a two-dimensional steady flow of an ideal fluid in a stream of finite depth, disturbed by some bottom topography $\hat{y} = \hat{B}(\hat{x})$. Upstream of the disturbance, the flow is uniform with depth $h$ and speed $U$. The free surface of the fluid $\hat{y} = \hat{\eta}(\hat{x})$ is initially unknown and computed as part of the solution.

The assumption of a steady, two-dimensional flow of an ideal fluid allows us to define a velocity potential $\phi$ and requires that we solve Laplace’s equation

$$\nabla^2 \phi = 0 \quad (3.1)$$

subject to boundary conditions on the free surface and the stream bed.

We non-dimensionalise the problem with respect to the variables $U$
Figure 3.1: Diagram of the problem. Upstream of the obstruction, the flow is uniform with unit depth and speed. The obstruction on the stream bed is given by $y = B(x)$ and the unknown free surface by $y = \eta(x)$. 
and $h$. The flow is then uniform upstream with unit depth and speed, with the undisturbed free surface located at $y = 1$ and the unobstructed stream bed at $y = 0$. The dimensionless parameters of the flow are the dimensions of the obstruction and the non-dimensionalised flow rate, or Froude number,

$$F = \left( \frac{U^2}{gh} \right)^{\frac{1}{2}}. \quad (3.2)$$

A sketch of the non-dimensionalised problem can be seen in Figure 3.1. The problem is defined for arbitrary topography, but we have chosen to show two Gaussian shaped obstructions on the stream bed as this is one of the cases we will examine later.

There can be no flow normal to the surface of the fluid and also no flow normal to the stream bed, giving the conditions,

$$\eta'(x) = \frac{v}{u} \quad \text{on} \quad y = \eta(x), \quad (3.3)$$

$$B'(x) = \frac{v}{u} \quad \text{on} \quad y = B(x), \quad (3.4)$$

where $u$ and $v$ are the horizontal and vertical components of the fluid velocity respectively. We also have the condition of constant pressure on the free surface which gives, from the Bernoulli equation,

$$\frac{1}{2}F^2(u^2 + v^2 - 1) + \eta = 1 \quad \text{on} \quad y = \eta(x). \quad (3.5)$$

### 3.2 Linearised method

A solution to the linearised problem of an arbitrary shape on the stream bed is derived based on the work of Lamb [46] and applied to semi-elliptical and Gaussian-shaped obstructions. Similar linear solutions for different topography were computed by King and Bloor [44] and Forbes...
and Schwartz [26]. In particular a linear solution for flow over a semi-ellipse was computed by Forbes [17, 18] and is included here for completeness. We consider the non-dimensionalised problem defined in the previous section, but instead take the undisturbed free surface at $y = 0$ and the unobstructed stream bed at $y = -1$.

The Bernoulli equation then gives,

$$\frac{1}{2} F^2 (u^2 + v^2 - 1) + \eta = 0 \text{ on } y = \eta(x). \quad (3.6)$$

We have the conditions on the free surface and on the stream bed,

$$u \eta'(x) = v \text{ on } y = \eta(x), \quad (3.7)$$
$$u B'(x) = v \text{ on } y = B(x). \quad (3.8)$$

We assume that the obstruction is small, $B(x) = -1 + B_1(x)$, where $B_1(x)$ is small relative to the depth of the stream, and that the resulting disturbance is also small, $\phi = x + \Phi(x,y)$, where $\Phi(x,y)$ is a small perturbation to uniform flow. These assumptions allow us to obtain,

$$F^2 \frac{\partial \Phi}{\partial x} + \eta = 0 \text{ on } y = 0, \quad (3.9)$$
$$\eta'(x) = \frac{\partial \Phi}{\partial y} \text{ on } y = 0, \quad (3.10)$$

which combine to give,

$$F^2 \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial \Phi}{\partial y} = 0 \text{ on } y = 0 \quad (3.11)$$

and we also have the condition on the stream bed in the form

$$\frac{\partial \Phi}{\partial y} = B_1'(x) \text{ on } y = -1. \quad (3.12)$$

An appropriate solution to Laplace’s equation is

$$\Phi = \int_0^\infty [\kappa(k) \cosh ky + \omega(k) \sinh ky] \sin kx \, dk \quad (3.13)$$
and we can find the $\kappa(k)$ and $\omega(k)$ which satisfy the boundary conditions. Assuming that $B'_1(x)$ can be written in the form

$$B'_1(x) = \int_0^\infty \gamma(k) \sin kx \, dk$$  \hfill (3.14)

and using the conditions (3.11) and (3.12) we obtain

$$\kappa(k) = \frac{\gamma(k)}{F^2k^2 \cosh k - k \sinh k},$$  \hfill (3.15)

$$\omega(k) = \frac{F^2\gamma(k)}{F^2k \cosh k - \sinh k}.$$  \hfill (3.16)

Rearranging (3.9) to give the free surface elevation,

$$\eta(x) = -F^2 \frac{\partial \Phi}{\partial x} \text{ on } y = 0$$  \hfill (3.17)

and substituting $\Phi$ with the $\kappa(k)$ and $\omega(k)$ as given above, we obtain

$$\eta(x) = -\int_0^\infty \frac{\gamma(k) \cos kx}{k \cosh k - F^2 \sinh k} \, dk.$$  \hfill (3.18)

We can use this to evaluate the free surface flow over any obstruction $B_1(x)$ provided we can determine $\gamma(k)$ from equation (3.14) using an appropriate Fourier Sine Transform. In the following sections we will examine the two cases of semi-elliptical and Gaussian shaped obstructions on the stream bed.

### 3.2.1 Semi-elliptical obstructions

We consider a semi-ellipse of length $2\alpha$ and height $\beta$ on the stream bed by taking

$$B_1(x) = \begin{cases} \frac{\beta}{\alpha} \sqrt{\alpha^2 - x^2} & \text{for } -\alpha < x < \alpha \\ 0 & \text{otherwise.} \end{cases}$$  \hfill (3.19)
Then

\[
B'_1(x) = \begin{cases} 
  -\frac{\beta}{\alpha} x (\alpha^2 - x^2)^{-\frac{1}{2}} & \text{for } -\alpha < x < \alpha \\
  0 & \text{otherwise},
\end{cases}
\]

and we use an appropriate Fourier Sine Transform to obtain \( \gamma(k) \) in the form

\[
\gamma(k) = -\beta J_1(\alpha k)
\]

where \( J_1 \) is the Bessel function of the first kind of order one. The free surface elevation \( \eta \) is then given by

\[
\eta(x) = \beta \int_0^\infty \frac{J_1(\alpha k) \cos kx}{k \cosh k - F^{-2} \sinh k} \, dk.
\]

When the flow is supercritical \( F > 1 \), we can evaluate this integral numerically, but when the flow is subcritical \( F < 1 \) there is a singularity in the integrand. To work around the singularity, we follow the work of Lamb [46] and take

\[
\int_0^\infty \frac{J_1(\alpha k) \cos kx}{k \cosh k - F^{-2} \sinh k} \, dk = \Re \left[ \int_0^\infty \frac{J_1(\alpha k) e^{ikx}}{k \cosh k - F^{-2} \sinh k} \, dk \right],
\]

and integrate around a closed contour in the complex plane \( \xi = k + im \), consisting of the real axis and a semi-circle of radius \( R \rightarrow \infty \) over the upper-half plane. We note that there are singularities on the axes, at \( \xi = \pm \kappa \) and \( \xi = i\omega_j \), \( j = 1, 2, 3, \ldots \), corresponding to the roots of the transcendental equation \( \tanh \kappa = \kappa F^2 \). Noting that the integrand is even, and using the residue theorem, we obtain

\[
\int_0^\infty \frac{J_1(\alpha k)e^{ikx}}{k \cosh k - F^{-2} \sinh k} \, dk = i\pi \Res(I,\kappa) + i\pi \sum_{j=1}^\infty \Res(I,i\omega_j),
\]
where
\[
\text{Res}(I, \kappa) = \frac{\kappa J_1(\alpha \kappa) e^{i\kappa x}}{\sinh \kappa(\kappa^2 + F^{-2} - F^{-4})},
\]
(3.25)
\[
\text{Res}(I, i \omega_j) = -\frac{i \omega_j I_1(\alpha \omega_j) e^{-i \omega_j x}}{\sin \omega_j(\omega_j^2 - F^{-2} + F^{-4})}
\]
(3.26)
and \(I_1\) is the modified Bessel function of the first kind of order one.

Taking the real part and substituting back into equation (3.22)
\[
\eta(x) = -\frac{2 \beta \pi \kappa J_1(\alpha \kappa) \sin \kappa x}{\sinh \kappa(\kappa^2 + F^{-2} - F^{-4})}
+ \beta \pi \sum_{j=1}^{\infty} \frac{\omega_j I_1(\alpha \omega_j) e^{-\omega_j x}}{\sin \omega_j(\omega_j^2 - F^{-2} + F^{-4})}
\]
for \(x > 0\),
(3.27)
and since \(\eta(x)\) is an even function
\[
\eta(x) = \frac{\beta \pi \kappa J_1(\alpha \kappa) \sin \kappa x}{\sinh \kappa(\kappa^2 + F^{-2} - F^{-4})}
+ \beta \pi \sum_{j=1}^{\infty} \frac{\omega_j I_1(\alpha \omega_j) e^{\omega_j x}}{\sin \omega_j(\omega_j^2 - F^{-2} + F^{-4})}
\]
for \(x < 0\),
(3.28)

The flow must be uniform upstream, and since the problem is linear
we can superpose a train of waves to cancel those upstream. This gives
us the free surface elevation for subcritical Froude numbers as;
\[
\eta(x) = -\frac{2 \beta \pi \kappa J_1(\alpha \kappa) \sin \kappa x}{\sinh \kappa(\kappa^2 + F^{-2} - F^{-4})}
+ \beta \pi \sum_{j=1}^{\infty} \frac{\omega_j I_1(\alpha \omega_j) e^{-\omega_j x}}{\sin \omega_j(\omega_j^2 - F^{-2} + F^{-4})}
\]
for \(x > 0\)
(3.29)
\[
\eta(x) = \beta \pi \sum_{j=1}^{\infty} \frac{\omega_j I_1(\alpha \omega_j) e^{\omega_j x}}{\sin \omega_j(\omega_j^2 - F^{-2} + F^{-4})}
\]
for \(x < 0\),
(3.30)
where \(\kappa\) is the positive real root of the transcendental equation \(\tanh \kappa = \kappa F^2\) and the \(\omega_j, j = 1, 2, 3, \ldots\) are the imaginary roots.

The first term of equation (3.29) gives the form of the downstream
waves. We note that the amplitude of these waves can only be zero when
\[ J_1(\alpha \kappa) = 0. \] Therefore the ellipse lengths \( \alpha \) for which waveless solutions exist are not evenly spaced and are dependent on the Froude number \( F \), but are independent of the ellipse height \( \beta \). This is in agreement with the results of Forbes [17, 18]. These ellipse lengths that result in waveless solutions are used for comparison with solutions to the fully nonlinear problem in Chapter 5.

### 3.2.2 Gaussian obstructions

We start with a single Gaussian obstruction on the stream bed of height \( \varepsilon \) by taking,

\[ B_1(x) = \varepsilon e^{-x^2}. \]  

(3.31)

Then

\[ B_1'(x) = -2\varepsilon xe^{-x^2} \]  

(3.32)

and we use an appropriate Fourier Sine Transform to obtain \( \gamma(k) \) in the form,

\[ \gamma(k) = -\frac{\varepsilon}{\sqrt{\pi}} ke^{-\frac{k^2}{4}}. \]  

(3.33)

The surface elevation \( \eta \) is then

\[ \eta(x) = \frac{\varepsilon}{\sqrt{\pi}} \int_0^\infty \frac{ke^{-\frac{k^2}{4}} \cos kx}{k \cosh k - F^{-2} \sinh k} dk. \]  

(3.34)

Using the same method as for the semi-ellipse in the previous section and following Lamb [46], we obtain the free surface elevation for subcritical Froude numbers;

\[ \eta(x) = -2\varepsilon \sqrt{\pi} \frac{k^2 e^{-\frac{k^2}{4}} \sin kx}{\sinh k (k^2 + F^{-2} - F^{-4})} \]

\[ + \frac{\varepsilon}{\sqrt{\pi}} \sum_{j=1}^{\infty} \frac{\omega_j^2 e^{\frac{\omega_j^2}{4}} e^{-\omega_jx}}{\sin \omega_j (\omega_j^2 - F^{-2} + F^{-4})} \text{ for } x > 0, \]  

(3.35)
\[ \eta(x) = \varepsilon \sqrt{\pi} \sum_{j=1}^{\infty} \frac{\omega_j^2 e^{\frac{x^2}{4}} e^{\omega_j x}}{\sin \omega_j (\omega_j^2 - F^{-2} + F^{-4})} \text{ for } x < 0, \quad (3.36) \]

where \( \kappa \) is the positive real root of the transcendental equation \( \tanh \kappa = \kappa F^2 \) and the \( \omega_j, j = 1, 2, 3, ... \) are the imaginary roots.

To consider the case of two Gaussian obstructions on the stream bed, we take the first obstruction centred at \( x = -b/2 \) and the second centred at \( x = b/2 \). We are dealing with linear solutions, so we can simply superpose one solution onto the other, which gives;

\[ \eta(x) = -2 \varepsilon \sqrt{\pi} \frac{\kappa^2 e^{-\frac{x^2}{4}} (\sin \kappa (x + \frac{b}{2}) + \sin \kappa (x - \frac{b}{2}))}{\sinh \kappa (\kappa^2 + F^{-2} - F^{-4})} \\
+ \varepsilon \sqrt{\pi} \sum_{j=1}^{\infty} \frac{\omega_j^2 e^{\frac{x^2}{4}} (e^{-\omega_j(x + \frac{b}{2})} + e^{-\omega_j(x - \frac{b}{2})})}{\sin \omega_j (\omega_j^2 - F^{-2} + F^{-4})} \text{ for } x > \frac{b}{2}. \quad (3.37) \]

The first term gives the form of the waves downstream of the second Gaussian obstruction. We notice that there are zero waves downstream of the second obstruction when \( \sin \kappa (x + \frac{b}{2}) + \sin \kappa (x - \frac{b}{2}) = 0 \), which occurs when the separation \( b = (2k + 1)\pi/\kappa, k = 0, 1, 2, ... \). We note that these values are independent of the height of the obstruction and also apply to obstructions of negative height, i.e. trenches. These separation values are evenly spaced, unlike the case of a semi-ellipse on the stream bed as considered in the previous section. The obstruction separation values that result in waveless solutions are included in Chapter 6, where they are compared with solutions to the fully nonlinear problem.
Chapter 4

Finite depth flows. Part II; Nonlinear problem formulation and numerical methods

In order to solve the full nonlinear problem, we use a variation on the method of Forbes [19] in which the problem was formulated in the physical plane using arclength along the free surface as the independent variable. This method has been successfully applied in a number of different applications involving free surface flows, see for example Dias and Vanden-Broeck [14, 16], Hocking and Forbes [38] and Hocking [34].

We define a complex potential $f(z) = \phi + i\psi$, where $\psi(x, y)$ is the streamfunction. We consider the non-dimensionalised flow with the undisturbed free surface located at $y = 1$ and the unobstructed stream bed at $y = 0$, as defined in Section 3.1. Hence the streamline $\psi = 1$ corresponds to the free surface, $y = \eta(x)$, and the streamline $\psi = 0$ corresponds to
CHAPTER 4. NONLINEAR PROBLEM FORMULATION

22

the stream bed, \( y = B(x) \).

The function \( f'(z) \) is analytic, so we can apply Cauchy’s integral formula for any fixed point on the boundary contour, \( z_0 \);

\[
f'(z_0) = \frac{1}{i\pi} \int_{\Gamma} \frac{f'(z)}{z - z_0} \, dz
\]

where \( f'(z) = u - iv \), \( u \) and \( v \) are the horizontal and vertical components of the fluid velocity respectively, and \( \Gamma \) is the closed contour consisting of the stream bed and the free surface, connected by vertical lines at \( x \to \pm \infty \). Taking the real part of this we obtain the equation for \( u_0 = u(x_0, y_0) \),

\[
u_0 = -\frac{1}{\pi} \int_{\Gamma} \frac{u[(y - y_0) - y'(x - x_0)] + v[(x - x_0) + y'(y - y_0)]}{(x - x_0)^2 + (y - y_0)^2} \, dx.
\]

(4.2)

The integral along the vertical lines at \( x \to \pm \infty \) makes no contribution, so that

\[
u_0 = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u[(B - y_0) - B'(x - x_0)] + v[(x - x_0) + B'(B - y_0)]}{(x - x_0)^2 + (B - y_0)^2} \, dx
\]

\[
- \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u[(\eta - y_0) - \eta'(x - x_0)] + v[(x - x_0) + \eta'(\eta - y_0)]}{(x - x_0)^2 + (\eta - y_0)^2} \, dx.
\]

(4.3)

In order to evaluate the integrals in (4.3), we truncate the domain at \( x = \pm x_L \) and discretise the problem by taking \( x_j = -x_L + 2x_L(j - 1)/(N - 1) \), \( j = 1, 2, ..., N \). We make an initial guess for each point on the free surface, \( \eta_j \) and the horizontal velocity at each point along the free surface and stream bed, \( u_{T_j} \) and \( u_{B_j} \) respectively, and evaluate equation (4.3) at each of the points \( (x_j, \eta_j) \) and \( (x_j, B_j) \), \( j = 1, 2, ..., N \). The vertical velocities, \( v_{T_j} \) and \( v_{B_j} \), are calculated using the conditions (3.3) and (3.4).

When evaluating \( u_{B_j} \), the first integral in (4.3) has a singularity at
$x = x_j$ which can be removed by adding and subtracting terms as follows,

$$u_{B_j} = -\frac{1}{\pi} \int_{-x_L}^{x_L} \frac{(u_B - u_{B_j})(\Delta B - B' \Delta x) + (v_B - v_{B_j})(\Delta x + B' \Delta B)}{\Delta x^2 + \Delta B^2} \, dx$$

$$+ \left[ \frac{u_{B_j}}{\pi} \arctan \frac{\Delta B}{\Delta x} \right]_{-x_L}^{x_L} - \left[ \frac{v_{B_j}}{2\pi} \log |\Delta x^2 + \Delta B^2| \right]_{-x_L}^{x_L}$$

$$- \frac{1}{\pi} \int_{x_L}^{-x_L} \frac{u_T((\eta - B_j) - \eta' \Delta x) + v_T(\Delta x + \eta'(\eta - B_j))}{\Delta x^2 + (\eta - B_j)^2} \, dx,$$

where $\Delta x = x - x_j$ and $\Delta B = B - B_j$. Similarly for $u_{T_j}$ on the free surface,

$$u_{T_j} = -\frac{1}{\pi} \int_{-x_L}^{x_L} \frac{u_B((B - \eta_j) - B' \Delta x) + v_B(\Delta x + B'(B - \eta_j))}{\Delta x^2 + (B - \eta_j)^2} \, dx$$

$$- \frac{1}{\pi} \int_{x_L}^{-x_L} \frac{(u_T - u_{T_j})(\Delta \eta - \eta' \Delta x) + (v_T - v_{T_j})(\Delta x + \eta' \Delta \eta)}{\Delta x^2 + \Delta \eta^2} \, dx$$

$$+ \left[ \frac{u_{T_j}}{\pi} \arctan \frac{\Delta \eta}{\Delta x} \right]_{-x_L}^{-x_L} - \left[ \frac{v_{T_j}}{2\pi} \log |\Delta x^2 + \Delta \eta^2| \right]_{-x_L}^{-x_L},$$

where $\Delta \eta = \eta - \eta_j$.

We find the error in these integral equations, (4.4) and (4.5) along with the error in the Bernoulli equation (3.5) and iterate using Newton’s method to find the free surface elevation $y = \eta(x)$ at $j = 1, 2, ..., N$. The bottom surface $B(x)$ and its derivative $B'(x)$ are specified in advance while the remaining derivatives are approximated using centred finite differences and the integrals approximated using the trapezoidal rule.

In theory, we can use this method to find solutions for flow over any topography, and we consider two cases in the following two chapters. In Chapter 5 we examine the case of a semi-ellipse on the stream bed of length $2\alpha$ and height $\beta$, given by

$$B(x) = \begin{cases} 
\frac{\beta}{\alpha} \sqrt{\alpha^2 - x^2} & \text{for } -\alpha < x < \alpha \\
0 & \text{otherwise}, 
\end{cases}$$

(4.6)
and in Chapter 6 we examine Gaussian obstructions of height $\varepsilon$ and separation $b$, given by

$$B(x) = \varepsilon e^{-(x-b/2)^2} + \varepsilon e^{-(x+b/2)^2}.$$  \hspace{1cm} (4.7)

We wish to find the ellipse lengths and Gaussian separation distances that result in zero waves downstream. These can be found using the above method if we instead discretise the domain by

$$x_j = -x_L + 2x_L(j - 1)/(2N - 1), \ j = 1, 2, ..., 2N$$

and specify symmetry, i.e. $\eta(x) = \eta(-x)$. We make an initial guess for $\eta_j$, $u_{T_j}$, $u_{B_j}$, $j = 1, 2, ..., 2N$, as well as the obstruction length $\alpha$ in the ellipse case or obstruction separation $b$ for the Gaussian case. Fixing $\eta_1$, so we still have $3N$ equations in $3N$ unknowns, we iterate on the first $N$ points and force symmetry about $x = 0$ at the beginning of each iteration.

The problem was programmed in Fortran and, in general, solutions with waves were graphically reproducible with $N = 800$ points across the free surface from $-x_L$ to $x_L$ with a truncation as large as $x_L = 30$. A truncation of $x_L = 30$ allowed for a longer obstruction, or larger separation between obstructions. Increased accuracy could be achieved by decreasing $x_L$ when obtaining solutions for obstructions with shorter length or separation. When finding waveless solutions, we could use the symmetry and only required $N = 400$ points to achieve the same level of accuracy as this corresponded to $2N = 800$ points across the full length of the free surface.
Chapter 5

Finite depth flows. Part III;
Nonlinear problem of flow past a semi-ellipse

Here we present the results to the fully nonlinear problem of flow past a semi-ellipse using the numerical method described in the previous chapter.

Solutions obtained using the numerical method were graphically reproducible with values of $N$ as small as $N = 200$, but most of the results in the following section were obtained using $N = 400$ points, corresponding to $2N = 800$ points across the free surface from $-x_L$ to $x_L$. The numerical method converged quite well, but it was found to be improved by allowing the value of the truncation point $x_L$ to vary slightly. While this did not noticeably affect the appearance and parameter values of the final solutions, it did greatly improve the convergence behaviour of the numerical method. A truncation point of $x_L > 15$ was found to be sufficient in most cases but as the obstacle height increased and the waves
became longer it was necessary to increase this value to around $x_L = 30$
to obtain converged, consistent values. Comparison with the method of
Forbes [18] gave excellent agreement.

The content of the following section has been published in the paper
A note on waveless subcritical flow past a submerged semi-ellipse [39] and
is the result of joint work with the co-authors of the paper.

5.1 Results

Linear waveless solutions will correspond to the zeros of the Bessel func-
tion term, i.e. when $J_1(\alpha \kappa) = 0$. These values are asymptotically valid
as the object height approaches zero and will be useful for comparison
with the numerically computed solutions. They are shown as circles on
the horizontal axes of Figures 5.2 and 5.5.

To find the nonlinear waveless solutions numerically, we started with
an initial guess of uniform flow and a flat free surface located at $y = 1$
and used this to find a solution for a small obstruction. We used the
linear solution as a starting guess for the length of the semi-ellipse, and
in most cases a starting waveless solution was found quite quickly using
a guess of $\eta = 0$ with a small ellipse height. Further solutions were then
obtained by increasing the ellipse height in small steps using the solution
from the previous height as a new initial guess. In general we used the
length of the half-ellipse $\alpha$ as an unknown and fixed the height, $\beta$, but
in some cases, as the solution contours became horizontal, it was found
necessary to interchange these two variables.

It is important to note that in most situations there will be down-
stream waves on the free surface. Figure 5.1 shows a waveless solution
Figure 5.1: Results showing the surface shapes with $F = 0.5$, $\alpha = 1.6$ and three slightly different heights $\beta = 0.11, 0.1177, 0.12$. The solutions with waves computed for slightly greater and smaller ellipse heights reveal a change of phase as the ellipse height passes through the waveless case.
with two “normal” solutions at slightly higher and lower obstruction height. The waves usually generated can be seen changing phase as they pass through the waveless solution.

Due to the time required to perform the simulations in 1982, Forbes [17, 18] looked at only one value of Froude number and only considered part of the first four “harmonics” of the cancelling waves.

Here we considered a range of different Froude numbers and were able to go out to the 6th harmonic to see if the original behaviour was matched over a broader range of parameter values. Figure 5.2 shows the contours of waveless solutions for the case with Froude number $F = 0.5$, essentially repeating the work of Forbes [18]. The figure was computed with hundreds of points and consequently can be regarded as a continuous line rather than the set of points shown in Forbes [18]. The Roman numerals indicate the number of trapped waves above the ellipse. Forbes obtained essentially the first of these curves, labelled $I$ & $II$, and was able to compute solutions part of the way up branches labelled $III$ & $IV$ but was unable to compute the higher values of $\beta$ and the higher values of ellipse length, i.e. the higher modes. However, he did speculate that these curves would turn over and merge as the first two contours had done. This work then serves as both confirmation of the numerical method and of Forbes’ speculation. Furthermore, the higher modes also exhibit this behaviour. As we move to higher modes the maximum ellipse height that allows waveless solutions increases. The circles on the horizontal axis indicate the linear, waveless solutions. Clearly as the height tends to zero each branch approaches these points.

It is of interest to see how the solutions evolve as we move along the curve on which a single wave is trapped to that on which there are
Figure 5.2: Contour plot showing the values of semi-ellipse height $\beta$ and length $\alpha$ that result in waveless solutions for $F = 0.5$. The a, b and c refer to surface shapes shown in Fig. 5.3, while d, e, f and g to those on Fig. 5.4.
two waves. Figure 5.3 shows the evolution as we move along the curve in Figure 5.2 labelled as \( I \) & \( II \). As one proceeds upward on the left branch, the solution has just a single dip above the ellipse. As can be seen in Figure 5.3 the contour turns downward before the second dip begins to grow upward from the bottom of the initial dip. Even at the point indicated as ‘c’ on Fig. 5.2, there is still only a single dip. Moving downward to ‘b’ and then ‘a’, a wave appears and grows in relative magnitude. The transition from the single to double waves occurs between ‘b’ and ‘c’ rather than at the top of the arch where one might expect.

These results are matched in the double branch labelled \( III \) & \( IV \); see Figure 5.4, where the surface corresponding to the labelled points are shown. As we move up along the \( IV \) branch the central dip begins to disappear so that the surface shape morphs into those on the \( III \) branch. Again this occurs before the maximum height of the contour is reached and the transition occurs between ‘f’ and ‘g’. The other double branches exhibit similar behaviour. The curve marked VII and VIII also loops around suggesting that higher harmonics will also link up. Computations of higher loops are very time consuming as they require a much larger truncation to deal with the longer ellipse and hence require a larger number of points to maintain the same level of accuracy in the solutions. It is unlikely that these higher loops will reveal any different behaviour.

Calculations reveal that \( F = 0.5 \) is not the only value of Froude number for which this behaviour occurs. Figure 5.5 shows the first double contour for Froude numbers \( F = 0.4, 0.5, 0.6, 0.7, 0.8 \). It is interesting to note that the maximum height at which the waveless solutions exist
CHAPTER 5. FLOW PAST A SEMI-ELLIPSE

Figure 5.3: Surface shapes for $F = 0.5$ with $\beta = 0.05$, 0.10, 0.15 and corresponding $\alpha = 2.033$, 2.387, 2.814 on the branch denoted $I$ & $II$. As $\beta$ increases the double dip slowly dies away leaving a single dip. The location of the curves are labelled on Fig. 5.2 as a ($\beta = 0.05$), b ($\beta = 0.10$) and c ($\beta = 0.15$).
Figure 5.4: Surface shape for $F = 0.5$ with $\beta = 0.05, 0.10, 0.15, 0.17$ and corresponding $\alpha = 3.774, 4.351, 5.254, 5.732$ on the branch denoted III & IV. As $\beta$ increases the central dip slowly dies away. The location of these four surface shapes are labelled on Fig. 5.2 as d ($\beta = 0.05$), e ($\beta = 0.10$), f ($\beta = 0.15$), g ($\beta = 0.17$).
decreases as $F$ increases, and also that as $F$ increases the curves lean further and further away from the vertical. As $F$ increases the ellipse length increases quite dramatically. The effect of nonlinearity for $F = 0.8$ is quite extreme, but the nonlinear superposition effect is still present.
5.2 Summary

We have computed waveless subcritical solutions for flow past a semi-ellipse, and produced contours in parameter space defining the obstruction dimensions at which waveless solutions exist. An interesting merging of contour pairs was observed, both confirming and extending the work of Forbes [18]. Each pair of waveless contours was found to merge near the maximum ellipse height, verifying Forbes’ [18] speculation that this would occur. In addition, this merging behaviour was found to occur over a range of Froude numbers, with contours calculated from $F = 0.4$ to $F = 0.8$.

An interesting array of free surface shapes were found, with a number of waves trapped above the ellipse. The free surface was observed to morph into different shapes as we tracked the solution contours through parameter space.
Chapter 6

Finite depth flows. Part IV;
Nonlinear problem of flow
past Gaussian topography

In this chapter we present the results to the fully nonlinear problem of flow past two Gaussian obstructions using the numerical method of Chapter 4.

For the computation of results in the following section, $N = 400$ points were used, corresponding to $2N = 800$ points across the free surface from $-x_L$ to $x_L$. Some test runs were done using $N = 600$ points which showed that the solutions computed using $N = 400$ points were converged to graphical accuracy. The numerical method converged well with the Gaussian shaped obstructions and no variation to the truncation point $x_L$ was required to improve convergence as was the case with the semi-elliptical obstructions in the previous chapter.
6.1 Results

In order to find the contours of waveless solutions, we started with an initial guess of uniform flow and a flat free surface located at $y = 1$ and used this to find a solution for a small obstruction. We considered two Gaussian obstructions and our initial guess for the separation was that found using the linear methods, $b = \frac{(2k+1)\pi}{\kappa}$, $k = 0, 1, 2, \ldots$ where $\kappa$ is the positive real root of the transcendental equation $\frac{\tanh \kappa}{\kappa} = F^2$. We then increased the height (or depth if we were finding the contours for the trenches) by small steps using the solution from the previous height as the new initial guess. The separation was recorded at each height and contours were plotted showing the height and separation at which waveless solutions occur.

6.1.1 Contour behaviour

Contours were plotted for six different Froude numbers, $F = 0.5$, $F = 0.55$, $F = 0.6$, $F = 0.7$, $F = 0.8$ and $F = 0.9$, and are included in Figures 6.1, 6.2, 6.3, 6.4, 6.5 and 6.6 respectively. In these figures each of the circles represents a waveless solution to the nonlinear problem whilst the dashed lines represent the separation values that result in waveless solutions in the linearised problem.

Considering the contours for obstructions of positive height, we observe that the first of the positive contours for the case $F = 0.5$ is fairly straight in the middle, with a slight curve near $\varepsilon = 0$ and starts to level off near the maximum height. This first contour seems to terminate when waves that can not be eliminated start to form on the surface both upstream and downstream of the object. As the Froude number is
Figure 6.1: Contour plot showing the values of obstruction height $\varepsilon$ and separation $b$ that result in waveless solutions for $F = 0.5$. 
Figure 6.2: Contour plot showing the values of obstruction height $\varepsilon$ and separation $b$ that result in waveless solutions for $F = 0.55$. 
Figure 6.3: Contour plot showing the values of obstruction height $\varepsilon$ and separation $b$ that result in waveless solutions for $F = 0.6$. 
Figure 6.4: Contour plot showing the values of obstruction height $\varepsilon$ and separation $b$ that result in waveless solutions for $F = 0.7$. 
Figure 6.5: Contour plot showing the values of obstruction height $\varepsilon$ and separation $b$ that result in waveless solutions for $F = 0.8$. 
Figure 6.6: Contour plot showing the values of obstruction height $\varepsilon$ and separation $b$ that result in waveless solutions for $F = 0.9$. 
increased, the first positive contour straightens near $\varepsilon = 0$ and the maximum height reached decreases. For higher Froude numbers, the first positive contour continues after levelling off, with a long horizontal section of contour at the maximum height, as observed for the $F = 0.8$ and $F = 0.9$ cases. For the included cases up to $F = 0.7$, the remaining positive contours are fairly straight, deviating from the linear values, until they curl over just before terminating. For the cases $F = 0.5$, $F = 0.55$ and $F = 0.6$, the tops of these positive contours curl towards $b = 0$, while those for $F = 0.7$ curl away from $b = 0$. The second positive contour in the $F = 0.8$ and $F = 0.9$ cases also curls away from $b = 0$, but also extends further, approaching the horizontal section of the first contour, and in the $F = 0.9$ case, the second contour follows the horizontal path of the first contour, almost joining up.

The contours for obstructions of negative height (or trenches) have quite different behaviour. There is a negative contour which appears to continue from each of the positive contours, the first of which curves towards $b = 0$, while the remaining contours in all except the $F = 0.9$ cases form a “zig-zag” pattern, with each contour having a number of horizontal sections. There are also other contours present, which do not correspond to any of the positive contours and form loops that appear to originate from $b = 0$. These looped contours turn over at a depth corresponding with the turns at the beginning of each horizontal section in the second negative contour. As the Froude number is increased, the depths at which the first negative contour and the looped contours meet the $\varepsilon$-axis increase. Each of the horizontal sections of the contours also occurs at a larger depth, along with the tops of the looped contours. We note that the “zig-zag” type contours are present in the $F = 0.8$ case, but
Figure 6.7: Obstructions $B(x)$ of height $\varepsilon = 0.05$ for three different separations, $b = 1$, $b = 3$ and $b = 7$. Note that for $b = 1$ the two objects merge into a single obstruction.

are not included in the contour plot in Figure 6.5 due to the extensive computation times and similarity of the behaviour to the lower Froude number cases.

The $F = 0.9$ case has quite different behaviour for the negative contours, with the “zig-zag” behaviour not observed. Instead the second negative contour exhibits a gradual curve before becoming horizontal. These contours were only calculated out to a separation of $b = 20$, as extending these further would require a larger truncation and higher number of points for the numerical method to retain the same level of accuracy, which would result in significantly longer computation times.
We note that, particularly in the negative case, the contour behaviour is quite different at low separation values. The transition from two obstructions to one obstruction as the separation $b$ decreases, as observed in Figure 6.7, may explain the difference in the contour behaviour at these low separation values.

### 6.1.2 Solution behaviour; obstructions of positive height

The solutions for the free surface on the first of the positive contours have a dip above the obstruction that increases in depth as the obstruction increases in height. The solutions on the remaining contours have two dips, one above each obstruction, with a number of trapped waves in between. The solutions on the second positive contour have one half wave trapped between the obstructions, those on the third contour have one and a half trapped waves, those on the fourth have two and a half trapped waves, and so on. Some of the solutions from the first four positive contours for $F = 0.5$ can be seen in Figure 6.8. This behaviour seems to be consistent across the range of Froude numbers considered.

Some interesting behaviour was observed, however, in the $F = 0.9$ case, where the contours almost link up. The solutions on the first contour have a single trough above a single obstruction, but as the contour becomes horizontal and the obstructions separate, this trough in the free surface becomes broad and flat. The solutions from the second contour have one half wave trapped between two troughs, and as this contour becomes horizontal these solutions also widen, so that two wide troughs are observed with a broad half wave trapped in between. These solutions occur at almost the same obstruction height, with the height values dif-
Figure 6.8: Waveless solutions for $F = 0.5$ from the first four positive contours. (a) shows solutions from the first positive contour, (b) from the second positive contour, (c) from the third and (d) from the fourth. These solutions occur at values of $\varepsilon$ evenly spaced between the two values in each plot. Solutions in (a) have a dip directly above the obstruction, increasing in depth as the obstruction increases in height. Solutions in (b), (c) and (d) have a dip above each obstruction with a number of trapped waves in between.
Figure 6.9: Solutions from the horizontal sections of the positive contours for $F = 0.9$, with separation values $b = 14, 15, 16, 17, 18, 19, 20$. (a) shows solutions from the first contour, with obstruction height $\varepsilon = 0.0137$ and (b) shows solutions from the second contour, with obstruction heights increasing from $\varepsilon = 0.0114$ to $\varepsilon = 0.0135$. 
ferring only at the fourth decimal place at a separation of $b = 17$. Some of
the solutions from each contour are included in Figure 6.9 for the same
range of separation values. The solutions shown in Figure 6.9a are from
the horizontal section of the first positive contour and those in Figure 6.9b
are from the horizontal section of the second positive contour.

6.1.3 Solution behaviour; obstructions of negative
height

The solutions for flow over trenches differ significantly on the various
contours and their branches. We will first consider the solutions from
the negative contours which continue on from their corresponding pos-
itive contours. The first negative contour, for each of the Froude num-
bers considered, has solutions that have a single peak which increases in
height as the trench increases in depth. The remaining contours for all
cases except $F = 0.9$, which follow the “zig-zag” pattern, will be broken
down into “downward” and “horizontal” branches in order to describe
the behaviour of the solutions.

The first downward branch of each of the contours for the case $F = 0.5$
has solutions with two distinct peaks with no observable trapped waves
in between. This section ends at a very shallow depth and hence the
solutions calculated were very close to the turn at the bottom of each
branch and this may account for why no trapped waves were observed.
For the cases $F = 0.55$, $F = 0.6$, $F = 0.7$ and $F = 0.8$, the solutions
from the first downward section of each of the contours have two distinct
peaks which have a number of waves trapped between them which then
flatten out as the solutions become closer to the depth at which the
contour turns. For the $F = 0.55$ case, the second contour has no trapped
waves, the third contour has one half wave trapped between the peaks, the fourth has one and a half trapped waves and the fifth contour has two and a half trapped waves. As the Froude number is increased these trapped waves become greater in amplitude relative to the height of the peaks and also drop downward slightly so that the number of trapped waves increases. For the $F = 0.6$ and $F = 0.7$ cases, the second contour has one half wave trapped between the two peaks, the third contour has one and a half trapped waves, the fourth contour has two and a half trapped waves and the fifth has three and a half trapped waves. The included contours for the $F = 0.8$ case also exhibit the same behaviour as the $F = 0.7$ case. A comparison of solutions from the first section of the fourth negative contour for different Froude numbers is included in Figure 6.10.

In the four cases $F = 0.5$, $F = 0.55$, $F = 0.6$ and $F = 0.7$, the solutions from the first horizontal section of each of the “zig-zag” contours have two distinct peaks with a broad trough in between. This trough widens as the separation between the trenches is increased. Although not included on the contour plot, the first horizontal section of the second negative contour was calculated for the $F = 0.8$ case and the same solution behaviour was observed as for the lower Froude number cases. We note, however, that no lower branches were calculated and hence the $F = 0.8$ case is not included in the solution comparison for lower branches.

Solutions on the second downward branch of each of the “zig-zag” contours for the case $F = 0.5$ have two distinct peaks and develop a number of trapped waves in between the peaks which flatten out again by the turn at the bottom of the branch. The solutions on the second
Figure 6.10: Comparison of solutions from the first downward branch of the fourth negative contour. (a) shows solutions for $F = 0.5$, (b) for $F = 0.55$, (c) for $F = 0.6$ and (d) for $F = 0.7$. The values of $\varepsilon$ are evenly spaced between the two values in each plot. The solutions for $F = 0.5$ have two distinct peaks with no observable trapped waves in between whilst those for $F = 0.55$ have one and a half trapped waves. Solutions for $F = 0.6$ and $F = 0.7$ have two and a half trapped waves.
contour have no trapped waves, those on the third contour develop one half trapped wave and those on the fourth contour develop one and a half trapped waves. For the $F = 0.55$, $F = 0.6$ and $F = 0.7$ cases, the solutions on the second downward branch develop one half wave trapped at the top of each peak as the depth of the trenches is increased. The solutions also develop a number of waves trapped between the peaks which flatten out again by the bottom of the branch. The number of trapped waves is consistent across the three cases, $F = 0.55$, $F = 0.6$ and $F = 0.7$; the solutions on the second negative contour develop one half trapped wave, those on the third develop one and a half trapped waves and those on the fourth develop two and a half trapped waves. Figure 6.11 shows solutions from the second downward section of the second negative contour. We observe that for $F = 0.5$, the solutions have two distinct peaks, while those for $F = 0.55$ and $F = 0.6$ develop one half wave trapped at the top of each peak. Note that for $F = 0.7$, one half trapped wave also develops at the top of each peak, but this is not observed fully as the largest amplitude solution plotted corresponds with the deepest point on the contour plot and not with the bottom of the branch. We also observe that for larger $F$, the depth of the wave trapped at the top of the peak and the depth of the trough or trapped wave between the peaks are greater relative to the height of the peaks themselves.

Note that the contour plot for $F = 0.7$ does not include the branches past the second downward section so will not be included in the discussion of lower branches.

For the cases $F = 0.55$ and $F = 0.6$, the solutions from the second horizontal section of each of the “zig-zag” contours have two peaks with
Figure 6.11: Comparison of solutions from the second downward branch of the second negative contour. Values of $\varepsilon$ are evenly spaced between the two values indicated in each plot. (a) shows solutions for $F = 0.5$, (b) for $F = 0.55$, (c) for $F = 0.6$ and (d) for $F = 0.7$. Note that for $F = 0.7$, the largest depth solution plotted does not correspond with the bottom of the branch.
one half wave trapped at the top of each peak and a broad trough in between. This trough widens as the separation between the trenches is increased. The solutions for $F = 0.5$ differ in that they do not have any waves trapped at the top of the peaks.

The solutions on the third downward section of the “zig-zag” contours for the case $F = 0.5$ have a number of trapped waves between two peaks. These two peaks become narrow and distorted as the depth of the trenches is increased. The solutions on the second contour have no trapped waves between the peaks and those on the third contour develop one and a half trapped waves between the peaks which then flatten out again so that the solutions at the bottom of the branch have two peaks with a broad trough in between. For the cases $F = 0.55$ and $F = 0.6$, the solutions in the third downward section develop one trapped wave at top of each peak. Solutions on the second contour develop one half trapped wave between the two peaks and those on the third contour develop one and a half trapped waves. The third downward section of the fourth negative contour is only included in the case $F = 0.6$ and these solutions develop two and a half waves trapped between the peaks. A comparison of solutions from the third downward section of the second negative contour for the $F = 0.5$ and $F = 0.55$ cases is included in Figure 6.12.

We will now consider the behaviour of the solutions on the looped contours, which are those which do not appear to correspond with any of the positive contours and form loops that originate from the $\epsilon$-axis. These contours were observed in the $F = 0.5$, $F = 0.55$, $F = 0.6$ and $F = 0.7$ cases, but were not calculated for $F = 0.8$ or $F = 0.9$. To better explain these solutions we break these loops into “upper” and “lower” branches. The upper branch on each looped contour is taken from the
CHAPTER 6. FLOW PAST GAUSSIAN TOPOGRAPHY

Figure 6.12: Comparison of solutions from the third downward section of the second negative contour. (a) shows solutions for $F = 0.5$ and (b) for $F = 0.55$. The $\varepsilon$ values for the solutions in each plot vary evenly between the two values indicated.

shallower intercept with the $\varepsilon$-axis and followed upwards to the point at which the contour turns back on itself. The lower branch begins at this turn and follows the contour downwards to the deeper intercept with the $\varepsilon$-axis.

Included in Figure 6.13 are solutions from the looped contours for the $F = 0.5$ case. The solutions from the shallower loop are included with those from the upper and lower branches included in 6.13(a) and 6.13(b) respectively. The solutions from the upper and lower branches of the deeper loop are included in 6.13(c) and 6.13(d) respectively. We note that the highest amplitude solution in each of the plots corresponds with the deepest point of each branch. In (a), (b) and (c) this is the point at which the branch intercepts the $\varepsilon$-axis. In (d) the highest amplitude solution corresponds with the deepest point on the branch that was included in the contour plot. Due to the shape of the looped contours, the solutions in these plots are not evenly spaced in $b$ or $\varepsilon$, but are instead chosen to
CHAPTER 6. FLOW PAST GAUSSIAN TOPOGRAPHY

Figure 6.13: Solutions for $F = 0.5$ from the looped contours. The separation values for the solutions vary between the two $b$ values indicated. These are not evenly spaced values of $b$, but are chosen to show the evolution of the solutions as the separation changes. (a) shows solutions from the upper branch of the shallower contour and (b) shows solutions from the lower branch. (c) shows solutions from the upper branch of the deeper contour and (d) shows solutions from the lower branch. The highest amplitude solution in each of the plots corresponds with the deepest point on each branch.
best show the evolution of the solutions.

Following the shallower looped contour, starting from the $\varepsilon$-axis intercept of the upper branch, we observe that the solutions have a single peak which decreases in height and develops one half wave trapped at the top of the peak. Once the contour turns into the lower branch the peak increases in height and the half trapped wave flattens out. As we follow the contour further a narrow distorted peak develops before the branch intercepts the $\varepsilon$-axis. The solutions from the deeper contour differ in that the upper branch begins with solutions that have one half wave trapped at the top of a single narrow distorted peak. As we follow the upper branch away from the $\varepsilon$-axis the half trapped wave increases in depth and the peak decreases in height, increases in width and the distortion disappears. Once the contour turns into the lower branch the peak again increases in height, decreases in width, starts to become distorted and one and a half trapped waves start to develop. These trapped waves are not shown to develop fully as Figure 6.13 only includes the solutions corresponding to the points on the contour plot and not the full length of this branch.

Solutions from the looped contours for $F = 0.55$ are included in Figure 6.14. As before the solutions from the upper and lower branches of the shallower contour are included in 6.14(a) and 6.14(b) respectively and the solutions from the upper and lower branches of the deeper contour are included in 6.14(c) and 6.14(d) respectively.

We observe that the solutions from the shallower loop for the $F = 0.55$ case have similar behaviour to those from the deeper loop for the $F = 0.5$ case but without the distortion.

Following the deeper loop from the $\varepsilon$-axis intercept of the upper
Figure 6.14: Solutions for $F = 0.55$ from the looped contours. The values of the separation vary between the two $b$ values indicated on each plot. These $b$ values are not evenly spaced, but are instead chosen to best show the evolution of the solutions. (a) shows solutions from the upper branch of the shallower contour and (b) shows solutions from the lower branch. (c) shows solutions from the upper branch of the deeper contour and (d) shows solutions from the lower branch. The highest amplitude solution in each of the plots corresponds with the deepest point on the contour plot from the corresponding branch.
branch, we observe solutions with two and a half waves trapped at the
top of a single peak which decreases in height and increases in width.
The trough in the middle of the trapped waves increases in depth un-
til solutions are observed that have two peaks with with one half wave
trapped at the top of each peak and a shallow trough in between. Once
the contour turns into the lower branch the peaks increase in height, de-
crease in width and the trough between the two becomes shallower and
begins to develop one half trapped wave. Following the contour further
the solutions begin to develop into three and a half waves trapped at the
top of a single peak. Again these waves are not shown to develop fully as
only the solutions corresponding to the points included on the contour
plot are included in Figure 6.14.

The solutions from the looped contours for the cases $F = 0.6$ and
$F = 0.7$ are of similar form to those for the $F = 0.55$ case.

We now examine the $F = 0.9$ case separately. Although the solu-
tions on the first negative contour exhibit the same behaviour as the
lower Froude number cases, solutions on the second negative contour are
slightly different. The solutions on this contour have two peaks with a
half wave trapped in between. This half wave is quite large in amplitude
compared with the height of the peaks. As the contour turns and be-
comes horizontal, the half wave in the free surface widens and morphs
into a deep, broad trough. This behaviour can be observed in Figure 6.15,
where three solutions from the second negative contour are shown.
Figure 6.15: Solutions from the second negative contour for $F = 0.9$. The parameters for the three solutions shown from top to bottom are (a): $\varepsilon = -0.015$ and $b = 11.296$, (b): $\varepsilon = -0.0195$ and $b = 14.5$, and (c): $\varepsilon = -0.0197$ and $b = 19.9$. As the obstruction separation is increased, the half wave in the free surface between the obstructions develops into a broad trough.
6.2 Summary

We have calculated waveless subcritical solutions for Gaussian obstructions of both positive and negative height, further extending the work of Forbes [18]. Contours were plotted showing the values of obstruction height and separation at which waveless solutions exist. The contours for obstructions of positive height did not have any pairs that merged to form loops as was the case for the semi-ellipse in the previous chapter. However, in the higher Froude number cases, the positive contours followed a similar horizontal path through parameter space, almost linking up. Different free surface shapes were observed on each branch, suggesting the possibility of non-unique solutions.

Interesting behaviour was also observed for the obstructions of negative height, with many of the waveless contours forming a “zig-zag” type pattern with a number of horizontal sections. These horizontal sections meant that for these fixed trench depths, there could be waveless solutions for almost all separations greater than some particular value.

An extensive range of free surface shapes were calculated, with a number of waves trapped above the obstructions as well as between the obstructions. An interesting evolution of the shape of the free surface was observed as we tracked the various contours through parameter space.

We revisit this problem in the next chapter, by re-examining the higher Froude number cases using a weakly nonlinear analysis.
Chapter 7

Finite depth flows. Part V; Weakly nonlinear analysis

The Korteweg-de Vries (KdV) equation is a form of weakly nonlinear analysis and is regularly used in various problems in free surface fluid dynamics. The equation was derived by Korteweg and de Vries [45] and due to its simplicity can be a powerful tool in the analysis of free surface flows. The assumptions made in deriving the KdV equation imply that the results for the problem of flow over topography are valid for Froude numbers close to $F = 1$.

In this chapter we include a brief analysis of the suitability of the KdV equation for the problem of waveless subcritical flow over topography. Comparisons will be made between the waveless solutions to the linear, weakly nonlinear and fully nonlinear problems for different Froude numbers.
7.1 Weakly nonlinear problem

We take the problem as defined in section 3.1, but before we non-dimensionalise, we assume that the wavelength $L$ is much greater than the upstream depth $h$ and define a small parameter $\epsilon = (h/L)^2 \ll 1$. We then non-dimensionalise and expand the velocity potential, free surface and stream bed in powers of $\epsilon$. Full derivations of the KdV equation can be found in several papers, see for example [12, 25, 52].

We write the stationary forced Korteweg-de Vries equation

$$\frac{1}{6}\eta_{xxx} + \frac{3}{2}m_x - (F - 1)\eta_x = -\frac{1}{2}B_x$$

(7.1)

and integrating,

$$\eta_{xx} + \frac{9}{2}\eta^2 - 6(F - 1)\eta = -3B.$$  

(7.2)

Here the Froude number is defined as before, $F = U/(gh)^{1/2}$ and $B$ is the bottom topography. We consider the flow over two Gaussian obstructions of height $\epsilon$ and separation $b$, defined by

$$B(x) = \epsilon e^{-(x-b/2)^2} + \epsilon e^{-(x+b/2)^2},$$

(7.3)

to compare with the results of the previous chapter.

Normally a phase plane analysis is used when solving the KdV equation for this type of problem. See for example Binder et al. [5, 6], Pratt [52], Forbes and Hocking [25] and Dias and Vanden-Broeck [14, 15, 16]. We are only interested in subcritical solutions and the specific parameters that result in a waveless free surface. We only require a brief analysis of the suitability of this method so we use an equation solver in MATLAB based on a fourth and fifth order Runge-Kutta method to solve the forced KdV equation for a given bottom topography $B(x)$. 
Figure 7.1: Free surface profiles obtained using the KdV equation for $F = 0.9$ and $\varepsilon = 0.01$ with separations $b = 14.9, 14.95, 15, 15.05$ and $15.1$. We observe a change of phase in the waves as $b$ increases, with a waveless solution occurring at $b = 15$.

7.2 Results

In order to find the waveless solutions, we start with a fixed Froude number $F$, low obstruction height $\varepsilon$ and the obstruction separation $b$ that results in waveless solutions in the nonlinear case. We then slowly increase or decrease this separation $b$ until we find the point at which the waves change phase. An example of this phase change is included in Figure 7.1, where we observe five free surface profiles for fixed Froude number $F$ and obstruction height $\varepsilon$ with increasing separation $b$. We can then increase the obstruction height, using the waveless separation
Figure 7.2: Free surface profiles and phase plane diagrams obtained using the KdV equation for $F = 0.9$ and $\varepsilon = 0.01$ with separations $b = 14.5$ (a, b), $b = 15$ (c, d) and $b = 15.5$ (e, f).
value from the previous height as our new starting guess. Repeating this process we can track the waveless contours in parameter space and plot these alongside the linear and nonlinear contours. As the KdV equation is valid for Froude numbers close to $F = 1$, we start by comparing the contours for the $F = 0.9$ case, before comparing the contours for lower Froude numbers.

Although we are not using the phase plane analysis for this problem, we include an example here for completeness. In Figure 7.2 we include free surface profiles and their corresponding phase plane diagrams for a Froude number of $F = 0.9$, obstruction height $\varepsilon = 0.01$ and separation values of $b = 14.5$, $b = 15$ and $b = 15.5$. The waves change phase as we increase the obstruction separation, with a waveless solution occurring at $b = 15$. In the phase plane, the free surface will be waveless if the contour starts and finishes at the point $(0, 0)$ as shown in Figure 7.2d. If waves form the contour finishes on one of the unforced solution curves as shown in Figures 7.2b and 7.2f.

The phase plane analysis can be used in conjunction with an approximation of the obstructions with appropriate delta functions when the height of the obstruction is comparable to the length of its base [5, 16]. This is not the case with the Gaussian obstructions that we are examining here, nor the semi-elliptical obstructions considered in Chapter 5, as we would need to consider ellipses of height $\beta < 0.05$ and length $\alpha > 3$ in order to find waveless solutions at a Froude number of $F = 0.9$. Therefore we leave this type of analysis for future work with different shaped obstructions.

In Figure 7.3, we have contours in parameter space showing the obstruction heights and separations at which waveless solutions exist for a
Figure 7.3: Contour plot showing the values of obstruction height $\varepsilon$ and separation $b$ that result in waveless solutions for $F = 0.9$
Figure 7.4: A comparison of weakly nonlinear and fully nonlinear waveless solutions for $F = 0.9$ and $\varepsilon = 0.01$. (a) contains solutions from the first contour, with the weakly nonlinear solution occurring at the separation $b = 5.755$ and the fully nonlinear solution occurring at $b = 4.9626$. (b) contains solutions from the second contour, where the weakly nonlinear solution occurs at the separation $b = 15$ and the fully nonlinear solution occurs at $b = 13.042$. 
Froude number of $F = 0.9$. The first two contours for the linear, weakly nonlinear and fully nonlinear solutions are included for obstructions of both positive and negative heights. The vertical dashed lines represent the solutions to the linearised problem and are independent of the obstruction height. Each of the circles represents a solution to the full nonlinear problem and each of the stars represents a waveless solution to the KdV equation. At this high Froude number, the nonlinear effects are quite strong and the nonlinear contours quickly deviate from the linear values. The KdV contours follow the same general shape as the nonlinear contours, but are slightly offset. We also observe that the KdV curves do not approach the linear values as the obstruction height approaches zero, as the nonlinear curves do.

A comparison of weakly nonlinear and fully nonlinear waveless solutions is included in Figure 7.4. Each solution is taken from the respective contour for Froude number $F = 0.9$ and obstruction height $\varepsilon = 0.01$. In Figure 7.4a, we compare solutions from the first contour, with the weakly nonlinear waveless solution occurring at an obstruction separation of $b = 5.755$ and the fully nonlinear solution occurring at a separation of $b = 4.9626$. Solutions from the second contour are compared in Figure 7.4b, where the weakly nonlinear waveless solution corresponds to an obstruction separation of $b = 15$ and the fully nonlinear solution corresponds to a separation of $b = 13.042$. In each case the weakly nonlinear solution is of a similar shape to the fully nonlinear solution, with wider and deeper troughs. This can be partly attributed to the difference in separation values between the weakly nonlinear and fully nonlinear cases.

Comparisons of the first two waveless contours for the $F = 0.8$ and $F = 0.6$ cases are included in Figures 7.5 and 7.6 respectively. We
observe that for these lower Froude numbers, the KdV equation does not provide good agreement with the nonlinear results and do not agree with the linear results in the limit as the obstruction height approaches zero. This is somewhat expected, as the KdV equation is only valid for Froude numbers close to $F = 1$ for this problem. In the $F = 0.6$ case as shown in Figure 7.6, we note that the second KdV contour crosses the axis where the third linear and nonlinear curves would be (though not included). We also note that the nonlinear contours exhibit changing behaviour over the three cases, $F = 0.9$, $F = 0.8$ and $F = 0.6$, but the KdV contours all exhibit the same qualitative behaviour as the $F = 0.9$ case and do not follow the shape of the nonlinear contours for lower $F$ values.

Our interest in this problem of flow over topography is to find the parameter values that result in waveless solutions for a range of Froude numbers. The KdV equation does not offer the required level of accuracy in these parameters for Froude numbers lower than $F = 0.9$. However it does provide some qualitative properties of the wave behaviour. Future work would involve developing a method to use the forced KdV equation to automatically find waveless solutions. We would use it to calculate waveless contours and analyse solution behaviour for Froude numbers close to $F = 1$. This way we could extend the contours for the $F = 0.9$ case and closely examine the apparent linking of contours without the restriction of the long computation times that come with the numerical method for the fully nonlinear problem.
Figure 7.5: Contour plot showing the values of obstruction height $\varepsilon$ and separation $b$ that result in waveless solutions for $F = 0.8$
Figure 7.6: Contour plot showing the values of obstruction height $\varepsilon$ and separation $b$ that result in waveless solutions for $F = 0.6$. 
Chapter 8

Infinite depth flows.
Nonlinear problem of a line sink in a flowing stream

We now move on to the second major problem to be considered, that of a line sink that disturbs an otherwise uniform flowing stream. Here we are concerned with the regime in which the flow caused by the line sink is much stronger than the background flow of the stream as this has application in the selective withdrawal from reservoirs with a background flow. The flow due to a submerged sink in an otherwise stationary fluid has been studied extensively. In the infinite depth problem, solutions with a stagnation point on the free surface directly above the sink have been found to occur for Froude numbers in the region $0 < F_S < 1.42$ [35].

Forbes and Hocking [24] considered this problem with the inclusion of surface tension, which was found to have a significant effect on the solutions. The inclusion of small surface tension resulted in a large increase in the maximum Froude number at which stagnation point solutions ex-
ist. A non-uniqueness in the solutions was also found by increasing the deflection of the free surface.

In this chapter we aim to reproduce the results of Forbes and Hocking [24], before extending to include the added flow from upstream. We examine the general behaviour of the solutions and the maximum sink strength $F_S$ for a given non-dimensional upstream flow speed $\beta U$, before investigating possible reasons for solution breakdown. We then find non-unique solutions by increasing the deflection of the free surface.

### 8.1 Problem formulation

We will consider the two-dimensional steady flow of an ideal fluid past a sink in a stream of infinite depth. The sink is located at depth $H$ and is of strength $m$. Without the sink, the flow is uniform with speed $U$. The free surface of the fluid $y = \eta(x)$ is unknown and will be computed as part of the solution.

We immediately non-dimensionalise the problem with respect to the variables $m$ and $H$. The sink is then located at $y = -1$. A diagram of the non-dimensionalised problem can be seen in Figure 8.1. The dimensionless parameters are the stream flow relative to the sink strength,

$$\beta_U = \frac{UH}{m} \quad (8.1)$$

and the Froude number related to the sink strength,

$$F_S = \frac{m}{\sqrt{gH^3}}. \quad (8.2)$$

We note the relationship between these two parameters, $F_S \beta_U = U/\sqrt{gH}$ which gives us the Froude number relative to the strength of the stream, which we define as,
Figure 8.1: Diagram of the non-dimensionalised problem. The free surface is given by $y = \eta(x)$ and the sink is located at $x = 0, y = -1$. The relative strength of the sink is given by the Froude number $F_S = \frac{m}{\sqrt{gH^3}}$ and the relative strength of the upstream flow is given by $\beta_U = \frac{UH}{m}$. Note that this diagram is not to scale and is for demonstrative purposes only.
\[ F_U = \frac{U}{\sqrt{gH}} \] (8.3)

Our assumption of a steady, two-dimensional flow of an ideal fluid allows us to define a velocity potential \( \phi \) and requires that we solve Laplace’s equation

\[ \nabla^2 \phi = 0 \] (8.4)

subject to boundary conditions on the free surface. We have the condition of no flow normal to the surface of the fluid,

\[ v = (\beta_U + u)\eta'(x) \quad \text{on} \quad y = \eta(x), \] (8.5)

and the condition of constant pressure on the free surface, which we apply to the Bernoulli equation to give,

\[ \frac{1}{2} F_S^2 ((\beta_U + u)^2 + v^2) + \eta = \frac{1}{2} F_S^2 \beta_U^2 \quad \text{on} \quad y = \eta(x). \] (8.6)

### 8.2 Linearised problem

We assume a small surface deflection so that we linearise around the rigid-lid solution, as we will be working with a low upstream flow rate. The assumption of a small deflection of the free surface allows us to obtain the linearised free surface condition

\[ \phi_y = 0 \quad \text{on} \quad y = 0, \] (8.7)

and we define the additional upstream flow in the velocity potential, along with the sink and its image

\[ \phi_0 = \beta_U x - \frac{1}{2\pi} \log (x + i(y + 1)) - \frac{1}{2\pi} \log (x + i(y - 1)). \] (8.8)
We can then differentiate with respect to $x$, evaluate on $y = 0$ and substitute into equation (8.6). Rearranging, we then obtain the equation for the free surface

$$
\eta(x) = -\frac{F_S^2 x^2}{2\pi^2(x^2 + 1)^2} + \frac{\beta U F_S^2 x}{\pi(x^2 + 1)}. \quad (8.9)
$$

Figure 8.2 shows four solutions for the same Froude number $F_S = 0.5$ and increasing stream flow $\beta U = 0, 0.05, 0.1, 0.15$. The same scale has been used to clearly show the effects of the added stream. When there is no additional flow from upstream, $\beta U = 0$, the flow is symmetric about the sink, with a peak in the free surface directly above the sink and a trough either side. As the upstream flow is increased, the trough upstream of the sink becomes deeper and the trough downstream from the sink becomes shallower. As $\beta U$ is increased further, the free surface downstream of the sink starts to rise and the downstream trough disappears. At $\beta U = 0.15$, the free surface gradually drops into a low wide trough upstream of the sink and has a single wide peak downstream of the sink.

### 8.3 Nonlinear problem and numerical method

In order to consider the full nonlinear problem we define a complex potential $f(z) = \phi + i\psi$, where $\psi(x,y)$ is the streamfunction. We consider the non-dimensionalised flow with the undisturbed free surface located at $y = 0$ and the sink located at $x = 0, y = -1$.

The function $f'(z)$ is analytic, so we can apply Cauchy’s integral formula for any fixed point on the boundary contour, $z_0$;

$$
f'(z_0) = \frac{1}{i\pi} \int_{\Gamma} \frac{f'(z)}{z - z_0} dz \quad (8.10)
$$
Figure 8.2: Four linear solutions for $F_S = 0.5$, with $\beta_U = 0, 0.05, 0.1$ and 0.15 from top to bottom.
where \( f'(z) = u - iv \) and \( u \) and \( v \) are the horizontal and vertical components of the fluid velocity respectively. \( \Gamma \) is the closed contour consisting of the free surface, the semicircle \( z = R_1 e^{i\theta_1}, \ \theta_1 \in (-\pi, 0), \ R_1 \to \infty \), the circle around the sink, \( z = R_2 e^{i\theta_2}, \ \theta_2 \in (0, -2\pi), \ R_2 \to 0 \) and the vertical lines that connect this circle with the free surface. A sketch of the contour \( \Gamma \) is included in Figure 8.3.

We take the real part to give us the integral equation for \( u_0 = u(x_0, y_0) \),

\[
 u_0 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u[(y - y_0) - (x - x_0)y'(x)] + v[(x - x_0) + (y - y_0)y'(x)]}{(x - x_0)^2 + (y - y_0)^2} \, dx 
\]

\[
- \frac{x_0}{\pi(x_0^2 + (y_0 + 1)^2)}. \quad (8.11)
\]
The first term is the integral along the free surface and the last term has originated from the integral of the circle around the sink.

In order to evaluate the integral in (8.11), we truncate the domain at \( x = \pm x_L \) and discretise the problem by taking

\[
    x_j = -x_L + 2x_L(j - 1)/(N - 1),
\]

\( j = 1, 2, ..., N \). We make an initial guess for each point on the free surface, \( \eta_j \) and the horizontal velocity at each point along the free surface, \( u_j \), and evaluate equation (8.11) at each of the points \( (x_j, \eta_j) \), \( j = 1, 2, ..., N \).

The vertical velocities, \( v_j \), \( j = 1, 2, ..., N \), are calculated using the condition (8.5).

When evaluating \( u_j \), \( j = 1, 2, ..., N \), the integral in (8.11) has a singularity at \( x = x_j \) which can be removed by adding and subtracting terms as follows,

\[
    u_j = \frac{1}{\pi} \int_{-x_L}^{x_L} \frac{(u - u_j)((\eta - \eta_j) - (x - x_j)\eta'(x)) + (v - v_j)((x - x_j) + (\eta - \eta_j)\eta'(x))}{(x - x_j)^2 + (\eta - \eta_j)^2} \, dx
    - \frac{x_j}{\pi(x_j^2 + (\eta_j + 1)^2)} + \left[ \frac{u_j}{\pi} \arctan \frac{\eta - \eta_j}{x - x_j} \right]_{-x_L}^{x_L}
    + \left[ \frac{v_j}{2\pi} \log[(x - x_j)^2 + (\eta - \eta_j)^2] \right]_{-x_L}^{x_L}. \tag{8.12}
\]

Making an initial guess for \( \eta_j \) and \( u_j \), \( j = 1, 2, ..., N \), we find the error in this integral equation (8.12), along with the error in the Bernoulli equation (8.6) and iterate to find the free surface elevation \( y = \eta(x) \).

The derivatives are approximated using centred finite differences and the integrals approximated using the trapezoidal rule.

The problem can be modified to incorporate surface tension by adding a term to the Bernoulli equation as follows,

\[
    \frac{1}{2} F_S^2((\beta U + u)^2 + v^2) + \eta(x) = \frac{1}{2} F_S^2 \beta_U^2 + T \frac{\eta''(x)}{[1 + (\eta'(x))^2]^{3/2}} \tag{8.13}
\]
where $T$ is the coefficient of surface tension. The integral equation (8.11) and kinematic condition (8.5) remain unchanged.

For the computation of results in the following section, $N = 1000$ points were used across the free surface from $-x_L$ to $x_L$. Most of the results were obtained with $x_L = 20$, but some test runs were done with different $x_L$ values to check the convergence of the method.

In order to further verify this method we reproduced some of the results of Forbes and Hocking [24] with no added uniform stream flow, before extending these results to include flow from upstream.

### 8.4 Results

Solutions were computed up to a maximum Froude number $F_S$ for given stream flow values $\beta U$. We will first consider the general behaviour of solutions in section 8.4.1 before looking at non-unique solutions in section 8.4.2.

#### 8.4.1 General behaviour

Figure 8.4 shows the evolution of the steady free surface as the Froude number relative to the sink strength $F_S$ is increased, given a fixed stream flow of $\beta U = 0.08$ and surface tension $T = 0$. The depth of the main trough increases as expected and we also observe a relatively short, narrow peak develop downstream of this trough. The free surface is flat in the far field with no waves observed. We note that we do not expect waves to be visible here, due to the low flow rates being considered.

We compare a linear solution with a nonlinear solution of the same parameters, $F_S = 0.5$, $\beta U = 0.2$ and $T = 0$, in Figure 8.5. We observe
Figure 8.4: Evolution of the steady free surface as $F_S$ is increased, given $\beta_U = 0.08$ and $T = 0$. The solutions shown are evenly spaced in $F_S$ from $F_S = 0.1$ to $F_S = 1$.

Figure 8.5: A comparison of linear and nonlinear solutions for $F_S = 0.5$, $\beta_U = 0.2$ and $T = 0$. The nonlinear solution is shown by the solid line and the linear solution is shown by the dashed line.
CHAPTER 8. A LINE SINK IN A FLOWING STREAM

Figure 8.6: Solutions for different surface tension values $T = 0$, $T = 0.01$ and $T = 0.02$, with $\beta_U = 0.2$, $F_S = 0.6$.

that the linear and nonlinear solutions provide good agreement, with the linear solution having a slightly shallower trough than the nonlinear solution.

A comparison of solutions with different surface tension values is shown in Figure 8.6. The three solutions were computed using the same stream flow $\beta_U = 0.2$ and Froude number $F_S = 0.6$, with different surface tension values of $T = 0$, $T = 0.01$ and $T = 0.02$. As observed in Figure 8.6, the three solutions are almost identical, with only a slightly shorter peak and shallower trough in the solutions with higher surface tension. These surface tension values are quite small, and larger values will have a greater effect on solutions as shown by Forbes and Hocking [24]. We only include these small surface tension values as they are used for comparisons in later figures.

Figure 8.7 shows a solution for a Froude number $F_S = 1.1$, stream flow $\beta_U = 0.21$ and surface tension of $T = 0.01$. This Froude number $F_S = 1.1$ is the maximum at which steady solutions exist for these given
\( F_S = 1.1, \beta_U = 0.21, T = 0.01 \). We observe that this solution has a narrowing in the trough above the sink. This effect is only observed in solutions with non-zero surface tension and nearing the maximum \( F_S \).

Solutions were computed up to the maximum Froude numbers for given stream flow values \( \beta_U \). The maximum attainable Froude numbers \( F_S \) were recorded and have been included in Figure 8.8a. Results obtained in terms of the stream Froude number \( F_U = F_S \beta_U \) are displayed in Figure 8.8b.

The maximum solutions with no surface tension, \( T = 0 \), were difficult to obtain. The numerical method used would compute solutions for Froude numbers above the maximum, but these would develop small numerically generated jagged waves as the Froude number increased past the maximum. The task then became identifying at which Froude numbers these numerically generated jagged waves began to develop. As a result of this, the \( T = 0 \) curves are not smooth.

The maximum solutions with surface tension were easier to obtain, with no jagged waves being observed for the surface tensions used, \( T = \)
Figure 8.8: Maximum parameters for which steady solutions exist. (a) shows the maximum sink Froude numbers, $F_S$ for given stream flows $\beta_U$. (b) shows the maximum sink Froude numbers, $F_S$ for given stream Froude number $F_U = F_S \beta_U$. 
0.01 and $T = 0.02$. Solutions were computed given $\beta U$ and $T$ for increasing $F_S$ until the numerical method failed to converge. The maximum values were taken from the last converged solutions. As a result, the curves for $T = 0.01$ and $T = 0.02$ in Figure 8.8 are much smoother.

We observe in Figure 8.8b, that the relationship between the maximum $F_S$ and $F_U$ appears to be linear. A line of best fit was calculated for each case;

\begin{align*}
F_S &= -6.792F_U + 1.491 \text{ for } T = 0, \quad (8.14) \\
F_S &= -5.347F_U + 2.340 \text{ for } T = 0.01, \quad (8.15) \\
F_S &= -5.197F_U + 2.626 \text{ for } T = 0.02, \quad (8.16)
\end{align*}

which we can use to estimate the curves for the maximum sink Froude number $F_S$ in terms of the stream flow $\beta U$, simply by substituting $F_U = F_S \beta U$ and rearranging. This suggests that, should we be able to estimate the equations for these lines for any value of surface tension, we would have a way of evaluating the parameters of the maximum steady flow without having to numerically generate solutions to the full problem. We leave this for future work.

For each of the solutions shown in Figure 8.8, the local maximum about the ‘peak’ and minimum of the ‘trough’ were recorded. The maxima and minima are shown in Figure 8.9. Figures 8.9a and 8.9b show the local maxima in terms of the stream flow values $\beta U$ and stream Froude number $F_U$ respectively, and Figures 8.9c and 8.9d show the minima in terms of the stream flow values $\beta U$ and stream Froude number $F_U$ respectively.

An approximation to the maximum slope was also calculated for each of the solutions, using the recorded maximum and minimum. These slope
Figure 8.9: Shows the height of the maximum ‘peak’, and the depth of the minimum ‘trough’ for given $\beta_U$ and $F_U$. The $\beta_U$ and $F_U$ values are those as shown in Figure 8.8.
Figure 8.10: Shows an approximation to the maximum slope of the free surface. This slope is calculated using the maximum and minimum values from Figure 8.9.
values are given in Figure 8.10, with the points shown corresponding to the solutions represented in Figure 8.8. Figure 8.10a shows the approximated maximum slope in terms of the stream flow values $\beta U$ and Figure 8.10b shows the approximated maximum slope in terms of the stream Froude number $F_U$. The magnitude of the slope decreases as we follow the solution curve in Figure 8.8a, with a sharp change observed at approximately $\beta U = 0.15$. This represents a distinct change in the free surface shape as the solutions evolve from a shape that is symmetric about the sink with two troughs, into the general shape observed in Figure 8.7. The exact maximum slope of each of the solutions was also calculated but has not been included as the slope values are quite small. We determine from these results that the solution breakdown at the maximum Froude number cannot be attributed to the slope becoming too large.

To further investigate possible reasons for solution breakdown, we examine the maximum curvature of the free surface. Solutions were computed up to the maximum sink Froude number $F_S$ for set stream flow $\beta U$ and surface tension $T$ values, and the maximum curvature recorded. This was done for stream flow values of $\beta U = 0$, $\beta U = 0.2$ and $\beta U = 0.4$, with surface tension values $T = 0$, $T = 0.01$ and $T = 0.02$. The results are displayed in Figure 8.11. We observe that the maximum curvature for solutions with surface tension rapidly increases as they approach the maximum Froude number, but this is not observed for the solutions with no surface tension. It appears that the reason that the solutions with surface tension break down at the maximum Froude number is due to the curvature of the solutions becoming too large.
Figure 8.11: The maximum curvature of the free surface for given stream flow $\beta_U$ and surface tension $T$ values.
8.4.2 Non-unique solutions

Following the work of Forbes and Hocking [24], we sought non-unique solutions. To do this we require an extra parameter, which we take as the total disturbance of the free surface, or deflection

$$D = \int_{-x_L}^{x_L} \sqrt{1 + \left(\frac{d\eta}{dx}\right)^2} - 1 \, dx. \quad (8.17)$$

We then allow the Froude number $F_S$ to be an unknown and increase the deflection $D$. We first sought to reproduce the results of Forbes and Hocking [24], and compute the curves with zero stream flow. This was done for surface tension values of $T = 0$, $T = 0.01$ and $T = 0.02$. These results were then extended and curves produced for solutions with stream flow values of $\beta U = 0.1$ and $\beta U = 0.2$ with surface tension values $T = 0$,
Figure 8.13: A comparison of two solutions for the same parameters, \( F_S = 1.094, \beta_u = 0.2 \) and \( T = 0.01 \).

\( T = 0.01 \) and \( T = 0.02 \). These results can be found in Figure 8.12. The curves representing the solutions with zero surface tension do not curl over, in agreement with the results of Forbes and Hocking [24]. The curves for solutions with surface tension values of \( T = 0.01 \) and \( T = 0.02 \) however do curl over, indicating a large range of Froude numbers at which two solutions exist. The Froude number at which each curve turns back on itself is the maximum for the given stream flow \( \beta_u \) and surface tension \( T \), and these agree with the values shown in Figure 8.8, as well as the asymptotes in Figure 8.11.

An example of two solutions with the same parameters can be seen in Figure 8.13. These solutions are for \( F_S = 1.094, \beta_u = 0.2 \) and \( T = 0.01 \), with different deflection values. The solution with the higher deflection has a deeper trough than the solution with the lower deflection and has also developed a secondary peak upstream of the sink. The solutions for higher deflection values are likely to be unstable mathematical solutions but this is not investigated here and will be the subject of further research.


8.5 Summary

We have successfully extended the work of Forbes and Hocking [24] to include an upstream flow past the sink. The results for no added stream agreed with those of Forbes and Hocking [24] and those with the added stream flow also demonstrated the same type of non-uniqueness in the solutions. In addition we have provided evidence to suggest that the breakdown of solutions with surface tension at the maximum Froude number can be attributed to the maximum curvature of the free surface becoming too large. In fact Figure 8.11 indicates a curvature singularity as the Froude number $F_S$ approaches the limiting value.

The addition of surface tension has allowed for a more thorough investigation into the characteristics of the solutions. Although the inclusion of small surface tension has little effect on the free surface shape for Froude numbers below the maximum for zero surface tension, it allows a much larger maximum Froude number and also allows for a much larger maximum curvature of the free surface.

No waves were found in our solutions apart from the jagged waves observed near the maximum Froude number for solutions with no surface tension, and these were assumed to not be real solutions and merely a consequence of the numerical method used.
Chapter 9

Conclusion

9.1 Finite depth flows over topography

In this thesis, we have examined the linearised, fully nonlinear and weakly nonlinear problems of flow over topography, focusing on waveless solutions. Solutions to the linearised problem were found by following the method of Lamb [46]. The fully nonlinear problem was formulated using an integral equation and solved using numerical methods. A weakly nonlinear analysis was performed using the Korteweg-de Vries equation which was also solved numerically.

We have calculated waveless subcritical solutions for flow over a semi-ellipse and also Gaussian obstructions of both positive and negative height, confirming and extending the work of Forbes [18]. In the semi-ellipse case, contours showing the ellipse height and length that produce waveless solutions were plotted for different Froude numbers. An interesting merging of solution branches occurs as the height of the ellipse increases at a constant value of the Froude number. In the case of Gaussian obstructions, the contours were plotted showing the obstruction height
and separation that result in waveless solutions for different Froude numbers. The contours did not merge together to form loops as was the case with the semi-ellipse, but an apparent linking of contours was present for Froude numbers close to $F = 1$. Some interesting behaviour was observed for the contours for obstructions of negative height, as once the trenches separated into two distinct holes the contours formed a kind of “ant farm” pattern, and in particular there were values of the trench depth for which there were waveless solutions for all separations greater than some particular value. For example, at $F = 0.6$ there exist waveless solutions for $\varepsilon \approx -0.35$ for all values of $b > 4$.

These waveless solutions are essentially an instance of a nonlinear superposition principle, with waves produced at each end of the ellipse, or by each Gaussian obstruction, effectively ‘cancelling’ as occurs in the linear theory [46].

The work presented here and in Forbes [18] indicate that these waveless flows can occur in a number of situations if the flow parameters are correctly adjusted. This leaves open the possibility to design undersea structures with significantly reduced drag by considering a range of different shapes and finding an optimal design for different ocean or river conditions.

## 9.2 Infinite depth flow past a line sink

In the last part of the thesis, we have considered the steady flow of an ideal fluid of infinite depth that is disturbed by a line sink. This was an extension of the work of Forbes and Hocking [24], who considered the flow due to a line sink with the inclusion of surface tension. They
found a non-uniqueness in the solutions by increasing the deflection of the free surface. We were able to reproduce the solutions of Forbes and Hocking [24], before including the added flow from upstream. The solutions with the added stream flow demonstrated the same type of non-uniqueness in the solutions as found by Forbes and Hocking [24].

A more thorough investigation into the characteristics of the solutions was possible with the inclusion of surface tension. A small surface tension value resulted in a larger maximum Froude number and a larger maximum curvature of the free surface, even though it had little effect on the shape of the free surface. No waves were observed in any of the real solutions, for any values of the surface tension. The investigation into the curvature of the free surface suggested that the breakdown of solutions with surface tension could be caused by the maximum curvature of the free surface becoming too large.

This problem has provided plenty of opportunities for further research. The unsteady version of the current problem would be of great interest, as this would allow us to investigate the stability of the non-unique solutions with higher deflection values. The finite depth version of the current problem would also be of interest, as waves have been observed in the finite depth problem of flow due to a line source or sink [37, 40, 49, 50, 63], and it would be interesting to see if solutions with waves can be found when the additional flow from upstream is present.

### 9.3 Concluding remarks

This thesis has examined two practical problems in free surface hydrodynamics. Analytical and numerical methods have been derived and utilised
to find solutions to the linear, weakly nonlinear and fully nonlinear problems. Some interesting behaviour has been discovered and investigated. As ever, many questions remain unanswered, and those that have been answered have raised others. These provide a fruitful source for future work.
Bibliography


BIBLIOGRAPHY


