A combinatorial approach to a problem in distribution modulo 1

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Séminaire du Laboratoire de MIS
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To completely describe the minimal intervals containing all the fractional parts $\{\xi 2^n\}, \ n \geq 0$, for some positive real number $\xi$. 
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**Main Tool:** Combinatorics on words
Words

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**Examples**

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- $(001)^\infty = 001001001001001001001001001001001001\ldots$

- $1100111100011011101111001101110010111111101\ldots$

- $100102110122220102110021111102212222201112012\ldots$

- $0123456789101112131415\ldots$

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- **Combinatorial group theory** involves the study of words that represent group elements.
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The extent to which a word exhibits strong regularity properties is generally inversely proportional to its “**complexity**”.

**Basic measure:** number of distinct blocks (factors) of each length occurring in the word.
Words: Factor Complexity

- Given a finite or infinite word $w$, let $\mathcal{F}_n(w)$ denote the set of distinct factors of $w$ of length $n \in \mathbb{N}^+$. 
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**Conjecture:** $C_x(n) = 2^n$ for all $n$ as it is believed $\sqrt{2}$ is normal in base 2.
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A few words about normal numbers

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It is widely believed that $\sqrt{2}$, $\pi$, and $e$ are normal in every base, but this conjecture is yet to be proved (or disproved).
Complexity & Periodicity

Theorem (Morse-Hedlund 1940)

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- Their low complexity accounts for many interesting features, as it induces certain regularities in such words without, however, making them periodic.
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- Numerous equivalent definitions & characterisations . . .
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- Words over a 2-letter alphabet \( \{a, b\} \) that are factors of (infinite) Sturmian words are called **finite Sturmian words** – they are the cyclic shifts of **Christoffel words** which can be obtained via the following construction.
Constructing Sturmian words

- Let’s consider a nice geometric realisation, starting with a special class of finite words . . .

- Words over a 2-letter alphabet \{a, b\} that are factors of (infinite) Sturmian words are called *finite Sturmian words* – they are the cyclic shifts of *Christoffel words* which can be obtained via the following construction.

- Using a similar construction we obtain *infinite Sturmian words*.
Christoffel words: Construction by example

Lower Christoffel word of slope $\frac{3}{5}$
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Lower & Upper Christoffel words of slope $\frac{3}{5}$

$L(3, 5) = aabaabab$  $U(3, 5) = babaabaa$
Sturmian words: Obtained *similarly* by replacing the line segment by a half-line:

\[ y = \alpha x + \rho \text{ with irrational } \alpha \in (0, 1), \rho \in \mathbb{R}. \]
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Example: \[ y = \frac{\sqrt{5}-1}{2} x \rightarrow \text{Fibonacci word} \]
From Christoffel words to Sturmian words

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\[ f = abaababaabaababaababaaba \cdots \text{ (note: disregard 1st } a \text{ in construction)} \]
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- \( f = abaababaabaababaaba \cdots \) (note: disregard 1st \( a \) in construction)
- **Standard Sturmian word** of slope \( \frac{\sqrt{5}-1}{2} \), golden ratio conjugate
Christoffel words: Historical notes

Before the 20th century:

- J. Bernoulli, 1771 (Astronomy)
- A. Markoff, 1882 (continued fractions)
- E. Christoffel, 1871, 1888 (Cayley graphs)
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Nowadays, they are mostly study in the context of Sturmian words.
### Examples

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<td>aababaababab</td>
</tr>
<tr>
<td>$U(p, q)$</td>
<td>bababa$a$</td>
<td>bbababa</td>
<td>babaabaabaa</td>
<td>bababaababaa</td>
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Properties

- $L(p, q) = awb \iff U(p, q) = bwa$
Christoffel words: Properties

**Examples**

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- $L(p, q) = awb \iff U(p, q) = bwa$
- $|L(p, q)|_a = q$, $|L(p, q)|_b = p \implies |L(p, q)| = p + q$
Christoffel words: Properties

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Properties

- $L(p,q) = awb \iff U(p,q) = bwa$
- $|L(p,q)|_a = q, |L(p,q)|_b = p \Rightarrow |L(p,q)| = p + q$
- $L(p,q)$ is the reversal of $U(p,q)$
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Properties

- $L(p, q) = awb \iff U(p, q) = bwa$
- $|L(p, q)|_a = q, |L(p, q)|_b = p \implies |L(p, q)| = p + q$
- $L(p, q)$ is the reversal of $U(p, q)$
- Christoffel words are of the form $awb, bwa$ where $w$ is a palindrome.
Theorem (folklore)

A finite word $w$ is a Christoffel word if and only if $w = aub$ or $w = bua$ where $u = \text{Pal}(v)$ for some word $v$ over $\{a, b\}$. 
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We define $Pal(\varepsilon) = \varepsilon$ (empty word)
Christoffel words & palindromes

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- Let $v^+$ denote the unique shortest palindrome beginning with $v$.

  We define $\text{Pal}(\varepsilon) = \varepsilon$ (empty word), and for any word $w$ and letter $x$,

  $$\text{Pal}(wx) = (\text{Pal}(w)x)^+.$$
Palindromic closure: Examples

- $(race)^+$ =
Palindromic closure: Examples

- \((race)^+ = race\)
Palindromic closure: Examples

\[(race)^+ = race\ car\]
Palindromic closure: Examples

- $(race)^+ = race\ car$
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- \((tops)^+ = top\ s\)
Palindromic closure: Examples

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- \(Pal(aba) = \)
Palindromic closure: Examples

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- \((tie)^+ = tie\ it\)
- \((tops)^+ = top\ spot\)
- \(Pal(aba) = \underline{a}\)
Palindromic closure: Examples

- \((\text{race})^+ = \text{race car}\)
- \((\text{tie})^+ = \text{tie it}\)
- \((\text{tops})^+ = \text{top spot}\)
- \(\text{Pal}(\text{aba}) = 
\underline{ab}\)
**Palindromic closure: Examples**

- \((race)^+ = race\ car\)
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- \(\text{Pal}(abc) = a \underline{b} a c \underline{a} b a\)

- \(\text{Pal}(race) = \underline{r}a \underline{c} \text{rar} \underline{e} \text{rar} \underline{e} \text{rar} \text{rar}\)
Palindromic closure: Examples

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- \(\text{Pal}(aba) = \underline{a}_b\underline{a}_a\underline{b}_a\)
- \(\text{Pal}(abc) = \underline{a}_b\underline{a}_c\underline{a}_b\underline{a}\)
- \(\text{Pal}(race) = \underline{r}_a\underline{r}_c\underline{r}_a\underline{r}_c\underline{r}_a\underline{r}_c\)
- \(L(3,5) = aabaabab = a\text{Pal}(aba)b\)
**Palindromic closure: Examples**

- \((race)^+ = race\ car\)
- \((tie)^+ = tie\ it\)
- \((tops)^+ = top\ spot\)

\[
\text{Pal}(aba) = \overbrace{a_b\ a\ a\ b\ a}\\
\text{Pal}(abc) = \overbrace{a_b\ a\ c\ a\ b\ a}\\
\text{Pal}(race) = \overbrace{r_a\ r_c\ r_a\ r_c\ r_a\ r_c}\\
\]

- \(L(3, 5) = aabaabab = a\text{Pal}(aba)b\)
- \(L(7, 4) = aabaabaabab = a\text{Pal}(abaa)b\)
Sturmian words: Palindromicity

**Theorem (de Luca 1997)**

An infinite word \( s \) over \( \{a, b\} \) is a standard Sturmian word if and only if there exists an infinite word \( \Delta = x_1x_2x_3 \cdots \) over \( \{a, b\} \) (not of the form \( ua^\infty \) or \( ub^\infty \)) such that

\[
s = \lim_{n \to \infty} \text{Pal}(x_1x_2 \cdots x_n) = \text{Pal}(\Delta).
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- **Example**: Fibonacci word is directed by $\Delta = (ab)(ab)(ab)\cdots$
Recall: Fibonacci word

Line of slope $\frac{\sqrt{5} - 1}{2} \rightarrow$ Fibonacci word
Recall: Fibonacci word

\[ \Delta = (ab)(ab)(ab) \cdots \rightarrow f = \text{Pal}(\Delta) = a \]
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Line of slope $\frac{\sqrt{5} - 1}{2} \rightarrow$ Fibonacci word

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Line of slope $\frac{\sqrt{5} - 1}{2} \rightarrow$ Fibonacci word

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Recall: Fibonacci word

\[ \Delta = (ab)(ab)(ab) \cdots \quad \rightarrow \quad f = Pal(\Delta) = abaa \]

Line of slope \( \frac{\sqrt{5}-1}{2} \) \( \longrightarrow \) Fibonacci word
Recall: Fibonacci word

\[ \Delta = (ab)(ab)(ab) \cdots \rightarrow f = Pal(\Delta) = \underline{aba}aba \]
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Line of slope $\frac{\sqrt{5} - 1}{2} \rightarrow$ Fibonacci word

$\Delta = (ab)(ab)(ab) \cdots \rightarrow f = Pal(\Delta) = \underline{aba}aba\underline{b}$
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Line of slope $\frac{\sqrt{5} - 1}{2} \rightarrow$ Fibonacci word

$\Delta = (ab)(ab)(ab) \cdots \rightarrow f = Pal(\Delta) = \underline{aba} \underline{aba} \underline{aba} \underline{ba} \cdots$
Recall: Fibonacci word

\[ \Delta = (ab)(ab)(ab) \cdots \longrightarrow f = Pal(\Delta) = \underline{aba} \underline{bab} \underline{aba} \underline{aba} \cdots \]

Note: Palindromic prefixes have lengths \((F_{n+1} - 2)_{n \geq 1} = 0, 1, 3, 6, 11, 19, \ldots\)
where \((F_n)_{n \geq 0}\) is the sequence of Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, 21, \ldots, defined by: \(F_0 = F_1 = 1, F_n = F_{n-1} + F_{n-2}\) for \(n \geq 2\).
A natural generalisation: Episturmian words

\{a, b\} \mapsto \mathcal{A} \ (\text{finite alphabet}) \text{ gives } \textit{standard episturmian words}.

\textbf{Theorem} \ (\text{Droubay-Justin-Pirillo 2001})

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r = Pal(\Delta) = a
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Amy Glen  (Murdoch University)  Combinatorics & Distribution Mod 1  July 12, 2011  35 / 49
A natural generalisation: Episturmian words

\{a, b\} \rightarrow \mathcal{A} (finite alphabet) gives standard episturmian words.

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\[r = Pal(\Delta) = ab\]
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\]

Example: \(\Delta = (abc)(abc)(abc) \cdots\) directs the Tribonacci word:

\[
\mathbf{r} = \text{Pal}(\Delta) = \underline{abac}
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Example: \( \Delta = (abc)(abc)(abc) \cdots \) directs the **Tribonacci word**:

\[
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Example: $\Delta = (abc)(abc)(abc) \cdots$ directs the \textit{Tribonacci word}:

$$r = \text{Pal}(\Delta) = \underline{abacaba} \underline{abacaba} \underline{abacaba} \underline{abacaba}$$
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\[
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\]

Note: Palindromic prefixes have lengths \(((T_{n+2} + T_n + 1)/2 - 2)_{n \geq 1} = 0, 1, 3, 7, 14, 27, 36\ldots\) where \((T_n)_{n \geq 0}\) is the sequence of \textit{Tribonacci numbers} 1, 1, 2, 4, 7, 13, 24, 44, \ldots, defined by:

\[
T_0 = T_1 = 1, \ T_2 = 2, \ T_n = T_{n-1} + T_{n-2} + T_{n-3} \quad \text{for} \ n \geq 3.
\]
Remarks

- Sturmian words, Christoffel words, and the $Pal$ operator all play important roles in the solution to the problem of interest.
Remarks

- Sturmian words, Christoffel words, and the $Pal$ operator all play important roles in the solution to the problem of interest.

- Before stating our main theorem, I will now discuss some background & motivation for the problem . . .
Some background & motivation

- Mahler (1968): defined the set of \( \mathcal{Z} \)-numbers by

\[
\mathcal{Z} := \left\{ \xi \in \mathbb{R}^+ \left| \forall n \geq 0, \ 0 \leq \left\{ \xi \left( \frac{3}{2} \right)^n \right\} < \frac{1}{2} \right\}
\]

where \( \{z\} \) denotes the fractional part of \( z \).
Some background & motivation

- **Mahler (1968):** defined the set of $\mathbb{Z}$-numbers by
  
  $$
  \mathcal{Z} := \left\{ \xi \in \mathbb{R}^+ \mid \forall n \geq 0, \ 0 \leq \left\{ \xi \left(\frac{3}{2}\right)^n \right\} < \frac{1}{2} \right\}
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  where $\{z\}$ denotes the fractional part of $z$.

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- It is still an open problem to prove that \( Z \) is in fact **empty**!

- A more general question:

  Given a real number \( \alpha > 1 \) and an interval \( (x, y) \subset (0, 1) \), does there exists \( \xi > 0 \) such that all fractional parts \( \{\xi \alpha^n\} \), \( n \geq 0 \), lie in the interval \( [x, y) \) or \( [x, y] \)?
Some background & motivation . . .

### Flatto-Lagarias-Pollington (1995)

If $p, q$ are coprime integers with $p > q \geq 2$, then any interval $(x, y)$ containing all fractional parts $\{\xi(p/q)^n\}$, $n \geq 0$, for some $\xi \in \mathbb{R}^+$ must satisfy $y - x \geq 1/p$. 
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Bugeaud-Dubickas (2005): described all irrational numbers $\xi > 0$ such that for a fixed integer $b \geq 2$ the fractional parts $\{\xi b^n\}, n \geq 0$, all belong to an interval of length $1/b$. 
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**Bugeaud-Dubickas (2005)**

Let $b \geq 2$ be an integer and $\xi > 0$ be an irrational number. Then:

- the frac. parts $\{\xi b^n\}$ cannot all lie in an interval of length $< 1/b$;
- there exists a closed interval of length $1/b$ containing the frac. parts $\{\xi b^n\}$ for all $n \geq 0$ iff the base $b$ expansion of the fractional part of $\xi$ is a Sturmian sequence on $\{k, k + 1\}$ for some $k \in \{0, 1, \ldots, b - 2\}$.
Fractional parts of powers & Sturmian words . . .

The core of Bugeaud & Dubickas’ result is the following fact that has been rediscovered several times in different contexts (since mid-late 80’s) . . .
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Notation

- Let $T$ denote the *shift map* on sequences: $T((s_n)_{n \geq 0}) := (s_{n+1})_{n \geq 0}$.  
  
  *$k$-th shift:* $T^k((s_n)_{n \geq 0}) := (s_{n+k})_{n \geq 0}$.  


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- Let $\preceq$ denote the *lexicographic order* on $\{0, 1\}^\mathbb{N}$ induced by $0 < 1$. 
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**Theorem**

If $s$ is a Sturmian word of (irrational) slope $\alpha > 0$ over the alphabet $\{0, 1\}$, then

$$0c_\alpha \preceq T^k(s) \preceq 1c_\alpha \quad \text{for all } k \geq 0,$$

where $c_\alpha$ is the (unique) standard Sturmian word of slope $\alpha$. 

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That is: all shifts of a Sturmian sequence $s \in \{0, 1\}^\mathbb{N}$ of slope $\alpha$ are lexicographically $\geq 0c_\alpha$ and lexicographically $\leq 1c_\alpha$. 
Example: Consider the Fibonacci word on \( \{0, 1\} \) (\( a \mapsto 0, b \mapsto 1 \)):

\[
f = 0100101001001010010\cdots,
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the standard Sturmian word \( c_\alpha \) of slope \( \alpha = (\sqrt{5} - 1)/2 \).
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\[
\begin{align*}
\underline{001001010010010100} & \prec \underline{001010010010010} \prec \underline{101001010010010100} \cdots \\
0f & \prec 1f
\end{align*}
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The main tool used by Bugeaud and Dubickas was combinatorics on words: replace real numbers by their base \( b \) expansions, and transform inequalities between real numbers into (lexicographic) inequalities between infinite sequences representing their base \( b \) expansions.
Given a sequence \( s \in \{0, 1\}^\mathbb{N} \), let \( r(s) \) denote the real number in \((0, 1)\) whose binary digits (after the binary point) are given by \( s \).
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r((01)_{\infty}) = \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots = \frac{1}{3}
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Given a sequence \( s \in \{0, 1\}^\infty \), let \( r(s) \) denote the real number in \((0, 1)\) whose binary digits (after the binary point) are given by \( s \).

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Let \( x, y \) be sequences on \( \{0, 1\} \). Then

\[ x \preceq y \quad \iff \quad r(x) \leq r(y). \]
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Also note that, for any sequence $s \in \{0, 1\}^\mathbb{N}$, $r(T^k(s)) = \{r(s)2^k\}$. 
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x \leq y \iff r(x) \leq r(y).
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So, for any given sequences $x, y, s$ over $\{0, 1\}$,

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$$(01)^\infty \preceq T^k((10)^\infty) \preceq (10)^\infty \quad \text{for all } k \geq 0$$

$$\frac{1}{3} \leq \left\{ \frac{2}{3} \cdot 2^k \right\} \leq \frac{2}{3} \quad \text{for all } k \geq 0$$
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Note: $R = 2/3$ is the smallest positive real number such that $[\frac{1}{3}, R]$ contains all the fractional parts $\{\xi2^k\}$, $k \geq 0$, for some $\xi \in \mathbb{R}^+$. 
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**Note:** \( R = 2/3 \) is the smallest positive real number such that \( \left[ \frac{1}{3}, R \right] \) contains all the fractional parts \( \{\xi 2^k\}, k \geq 0 \), for some \( \xi \in \mathbb{R}^+ \).

**Recall:** For any Sturmian word \( s \in \{0, 1\}^\mathbb{N} \) of slope \( \alpha \),

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0c_\alpha \preceq T^k(s) \preceq 1c_\alpha \quad \text{for all } k \geq 0,
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i.e.,

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r(0c_\alpha) \leq \{r(s)2^k\} \leq r(0c_\alpha) + \frac{1}{2} \quad \text{for all } k \geq 0.
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**Note:** $R = r(0c_\alpha) + \frac{1}{2}$ is the smallest positive real number such that $[r(0c_\alpha), R]$ contains all the fractional parts $\{\xi 2^k\}$, $k \geq 0$, for some $\xi \in \mathbb{R}^+$ (namely $\xi = r(s)$ where $s \in \{0,1\}^\mathbb{N}$ is a Sturmian word of slope $\alpha$).
Note: For $x \in \left[\frac{1}{2}, 1\right]$, there does not exist a real number $\xi > 0$ such that $x \leq \{\xi 2^k\} < 1$ for all $k \geq 0$ (Bugeaud-Dubickas with $b = 2$).
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**Definition**

Given a real number $x \in (0, 1/2)$, let $R(x)$ denote the smallest positive real number such that $[x, R(x)]$ contains all fractional parts $\{\xi 2^k\}, k \geq 0$, for some $\xi \in \mathbb{R}^+$. 
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Finally, we can state our theorem that gives a complete description of the minimal intervals containing all fractional parts $\{\xi 2^k\}$, $k \geq 0$, for some real positive real number $\xi \ldots$
Note: For $x \in \left[\frac{1}{2}, 1\right]$, there does not exist a real number $\xi > 0$ such that $x \leq \{\xi 2^k\} < 1$ for all $k \geq 0$ (Bugeaud-Dubickas with $b = 2$).

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Finally, we can state our theorem that gives a complete description of the minimal intervals containing all fractional parts $\{\xi 2^k\}$, $k \geq 0$, for some real positive real number $\xi$ . . .

For each $x \in (0, \frac{1}{2})$, the theorem gives the minimal interval beginning with $x$ (namely $[x, R(x)]$) that contains all fractional parts $\{\xi 2^k\}$, $k \geq 0$, for some $\xi \in \mathbb{R}^+$. 
**Theorem (Allouche & Glen)**

Let $x$ be a real number in $(0, \frac{1}{2})$.  

(i) If $(2x)_2$ is a standard Sturmian sequence, then

$$R(x) = x + \frac{1}{2}.$$
**Theorem (Allouche & Glen)**

Let \( x \) be a real number in \((0, \frac{1}{2})\).

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Furthermore, \( R(x) \) is the unique real number in \((0, 1)\) that has a Sturmian binary expansion and satisfies \( x \leq \{ R(x)2^k \} \leq R(x) \) for all \( k \geq 0 \).
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(ii) If $(2x)_2 = (Pal(v)01)\infty$ or $(2x)_2 = (Pal(v)10)\infty$ for some word $v$ over $\{0, 1\}$, then

$$R(x) = r((1Pal(v)0)\infty).$$
Theorem (Allouche & Glen)

Let $x$ be a real number in $(0, \frac{1}{2})$.

(i) If $(2x)_2$ is a standard Sturmian sequence, then

$$R(x) = x + \frac{1}{2}.$$  

Furthermore, $R(x)$ is the unique real number in $(0, 1)$ that has a Sturmian binary expansion and satisfies $x \leq \{R(x)2^k\} \leq R(x)$ for all $k \geq 0$.

(ii) If $(2x)_2 = (\text{Pal}(v)01)\infty$ or $(2x)_2 = (\text{Pal}(v)10)\infty$ for some word $v$ over \{0, 1\}, then

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(iii) In all other cases, $R(x) = r((1\text{Pal}(v)0)^\infty)$ where $v$ is the unique word over \{0, 1\} such that

$$r((\text{Pal}(v)01)^\infty) < 2x < r((\text{Pal}(v)10)^\infty).$$
Examples

Take $x = 1/4$.

- The **minimal interval** beginning with $x = \frac{1}{4}$ containing all the fractional parts $\{\xi 2^k\}$, $k \geq 0$, for some real number $\xi > 0$ is $[1/4, R(1/4)]$. 
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- Observe that $(2x)_2 = (1/2)_2 = 1000 \cdots$ (or 0111 \cdots).
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- Moreover, $\frac{1}{4} \leq \{\frac{2}{3} \cdot 2^k\} \leq \frac{2}{3}$ for all $k \geq 0$. 
Examples . . .

Let $x = 1/(\pi + \sqrt{2}e)$.

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$$ (x)_2 = \underbrace{010010}_{Pal(010)} 0101001010010001101101001100010001100 \cdots $$
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$$
R(x) = r((1\text{Pal}(010)0)^\infty) = 164/255.
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Remarks

- It is known (Ferenczi-Mauduit, 1997) that any real number having a Sturmian binary expansion is transcendental.
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As a consequence of our theorem, we deduce:

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• One may ask what happens with base $b$ expansions, where $b \geq 3$, or what can be said about the intervals containing all $\{\xi b^n\}$ for some $\xi$. 
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The result of Bugeaud and Dubickas (2005) implies that Sturmian sequences on an alphabet $\{k, k + 1\}$ for some $k \in \{0, 1, \ldots, b - 2\}$ will again play a fundamental role.
Merci pour votre attention!
References
