Rich, Sturmian & trapezoidal words

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Words

In mathematics, words naturally arise when one wants to represent elements from some set in a systematic way.
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- **Combinatorial group theory** involves the study of words that represent group elements.
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- Combinatorial group theory involves the study of words that represent group elements.

Formally:

- A word is a finite or infinite sequence of symbols (letters) taken from a non-empty countable set $\mathcal{A}$ (alphabet).
Examples

- 001
- \((001)^\infty = 001001001001001001001001001001\cdots\)
- 110011110001101110111001101110010111111101\cdots
- 100102110122220102110021111102212222201112012\cdots
- 0123456789101112131415\cdots
- 1121212121212\cdots
- 212114116118\cdots
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Examples

- $001$
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- \([2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots]\)
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Words . . .

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- Most commonly studied words are those which satisfy one or more strong regularity properties; for instance, words containing many repetitions or palindromes.

- The extent to which a word exhibits strong regularity properties is generally inversely proportional to its “complexity”.
  
  **Basic measure:** number of distinct blocks (factors) of each length occurring in the word.
Factor Complexity

Given a finite or infinite word $w$, let $\mathcal{F}_n(w)$ denote the set of distinct factors of $w$ of length $n \in \mathbb{N}^+$. 
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\mathcal{F}_3(x) = \{000, 001, 010, 100, 101, 110, 111\}, \quad C_x(3) = 8
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**Conjecture:** \( C_x(n) = 2^n \) for all \( n \) as it is believed \( \sqrt{2} \) is *normal* in base 2.
Complexity & Periodicity

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An infinite word $w$ is eventually periodic if and only if $C_w(n) \leq n$ for some $n \in \mathbb{N}^+$. 
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- Numerous equivalent definitions & characterisations . . .
Palindromic Complexity

Given a finite or infinite word $w$, let $P_w(n)$ denote the \textit{palindromic complexity function} of $w$, which counts the number of palindromic factors of $w$ of each length $n \geq 0$. 
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**Theorem (Droubay-Pirillo 1999)**

An infinite word $w$ is Sturmian if and only if

$$P_w(n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd} \end{cases}$$
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**Note:**

- Any Sturmian word is over a 2-letter alphabet since it has two distinct factors of length 1.
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- A Sturmian word over the alphabet $\{a, b\}$ contains either $aa$ or $bb$, but not both.
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Words over a 2-letter alphabet \{a, b\} that are factors of (infinite) Sturmian words are called *finite Sturmian words* – they are the cyclic shifts of *Christoffel words* obtained via the following construction.
Constructing Sturmian words

- Let’s consider a nice geometric realisation, starting with a special class of finite words . . .
- Words over a 2-letter alphabet \{a, b\} that are factors of (infinite) Sturmian words are called finite Sturmian words – they are the cyclic shifts of Christoffel words obtained via the following construction.
- Using a similar construction we obtain infinite Sturmian words.
Christoffel words: Construction by example

Lower Christoffel word of slope $\frac{3}{5}$
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$L(3,5) = aa$
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$L(3,5) = aab$
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Lower & Upper Christoffel words of slope $\frac{3}{5}$

$L(3,5) = aabaabab$  \quad U(3,5) = babaabaa
From Christoffel words to Sturmian words

**Sturmian words**: Obtained *similarly* by replacing the line segment by a half-line:

\[ y = \alpha x + \rho \text{ with irrational } \alpha \in (0, 1), \rho \in \mathbb{R}. \]
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- \( f = \text{abaababaabaababaaba} \cdots \) (note: disregard 1st a in construction)
- **Standard Sturmian word** of slope \( \frac{\sqrt{5} - 1}{2} \), golden ratio conjugate
Factor complexity of finite Sturmian words

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**Theorem (de Luca 1999)**

If \( w \) is a finite Sturmian word of length \( |w| \) (i.e., a cyclic shift of a Christoffel word), then the graph of \( C_w(n) \) as a function of \( n \) (for \( 0 \leq n \leq |w| \)) is that of a *regular trapezoid* (possibly degenerated to a triangle).
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That is:

- \( C_w(n) \) increases by 1 with each \( n \) on some interval of length \( r \).
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That is:

- $C_w(n)$ increases by 1 with each $n$ on some interval of length $r$.
- Then $C_w(n)$ is constant on some interval of length $s$.
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So if we set \( D_w(n) = C_w(n + 1) - C_w(n) \) for each \( n \) with \( 0 \leq n \leq |w| - 1 \), then the word \( D_w(0)D_w(1)\cdots D_w(|w| - 1) \) takes the form \( 1^r0^s(-1)^r \).
Example

Graph of the factor complexity of the Christoffel word $L(3, 5) = aabaabab$
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Trapezoidal words

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- \textbf{Note:} If \( w \) is a \textit{trapezoidal word} (i.e., its ‘complexity’ graph has the same behaviour as that of Sturmian words), then necessarily \( C_w(1) = 2 \).
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So any trapezoidal word is on a \textit{binary alphabet} and the family of trapezoidal words properly contains all finite Sturmian words.
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Characterisation of Sturmian palindromes

We have shown:

**Theorem (de Luca-G.-Zamboni)**

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Let $w$ be a trapezoidal word. Then $w$ contains $|w| + 1$ distinct palindromes (including $\varepsilon$).

That is, trapezoidal words (and hence finite Sturmian words) are “rich” in palindromes in the sense that they contain the maximum number of distinct palindromic factors since:
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That is, trapezoidal words (and hence finite Sturmian words) are “rich” in palindromes in the sense that they contain the maximum number of distinct palindromic factors since:

**Theorem (Droubay-Justin-Pirillo 2001)**
A finite word \( w \) contains at most \(|w| + 1\) distinct palindromes (including \( \varepsilon \)).
Rich words

Definition (G.-Justin 2007)
A finite word \( w \) is *rich* iff \( w \) contains exactly \( |w| + 1 \) distinct palindromes.
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**Examples**

- $abac$ is rich, whereas $abca$ is not rich.
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- $abac$ is rich, whereas $abca$ is **not** rich.
- The word **rich** is rich . . . and **poor** is rich too!
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- $a^\omega = aaaaaa \cdots$ and $ab^\omega = abbb \cdots$ are rich.
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Examples

- $a^\omega = aaaaaa \cdots$ and $ab^\omega = abbb \cdots$ are rich.
- $(ab)^\omega = abababab \cdots$ and $(aba)^\omega = ababaaba \cdots$ are rich.
- $abc$ is rich, but $(abc)^\omega = abcabcabc \cdots$ is not rich.
Roughly speaking, a finite or infinite word is rich if and only if a new palindrome is introduced at each new position.
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A characterisation of rich words

Let $u$ be a factor of a finite or infinite word $w$. 
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**Characteristic Property (G.-Justin 2007)**

A finite or infinite word $w$ is rich if and only if for each palindromic factor $p$ of $w$, every *complete return* to $p$ in $w$ is a palindrome.
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**Characteristic Property** (G.-Justin 2007)

A finite or infinite word $w$ is rich if and only if for each palindromic factor $p$ of $w$, every *complete return* to $p$ in $w$ is a palindrome.

In short, a word is rich if and only if all complete returns to palindromes are palindromes.
Rich words have appeared in many different contexts; they include:

- **Sturmian and episturmian words**
  - Droubay-Justin-Pirillo (2001)
  - Anne-Zamboni-Zorca (2005)

- **Complementation-symmetric Rote sequences**

- **Symbolic codings of trajectories of symmetric interval exchange transformations** – Ferenczi-Zamboni (2008)

- **A certain class of words associated with $\beta$-expansions where $\beta$ is a simple Parry number**

- **Infinite words with “abundant palindromic prefixes”**
  - Introduced by Fischler in 2006 in relation to Diophantine approximation
A Connection Between Palindromic & Factor Complexity

Allouche-Baake-Cassaigne-Damanik, 2003: for any aperiodic infinite word $w$,

$$P_w(n) \leq \frac{16}{n} C_w\left(n + \left\lfloor \frac{n}{4} \right\rfloor\right) \quad \text{for all } n \in \mathbb{N}.$$
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$$P_w(n) + P_w(n + 1) \leq C_w(n + 1) - C_w(n) + 2 \text{ for all } n \in \mathbb{N}. \quad (*)$$
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Bucci-De Luca-G.-Zamboni, 2008: infinite words $w$ for which $P_w(n) + P_w(n+1)$ reaches the upper bound in $(*)$ for every $n$ are rich . . .
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**Theorem** (Bucci-De Luca-G.-Zamboni 2008)

For any infinite word $w$ with set of factors $\mathcal{F}(w)$ closed under reversal, the following conditions are equivalent:

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Complementation-symmetric Rote sequences:

- Infinite words over $\{a, b\}$ with factors closed under both complementation and reversal, and such that $C(n) = 2n$ for all $n$. 
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- Infinite words over \( \{a, b\} \) with factors closed under both complementation and reversal, and such that \( C(n) = 2n \) for all \( n \).
- Allouche-Baake-Cassaigne-Damanik (2003): \( P(n) = 2 \) for all \( n \).
- Hence \( P(n) + P(n+1) = 4 = C(n+1) - C(n) + 2 \) for all \( n \) \( \Rightarrow \) RICH.
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Sturmian words:

- Morse-Hedlund (1940): $C(n) = n + 1$ for all $n$
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- **Morse-Hedlund (1940):** $C(n) = n + 1$ for all $n$

- **Droubay-Pirillo (1999):** $P(n) = 1$ for $n$ even, $P(n) = 2$ for $n$ odd
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- Hence $P(n) + P(n + 1) = 3 = C(n + 1) - C(n) + 2$ for all $n \Rightarrow$ RICH.
Finite Case

Using completely different methods . . .

**Theorem (de Luca-G.-Zamboni)**

For any finite word $w$, the following two conditions are equivalent:

i) $w$ is a rich palindrome;

ii) $P_w(n) + P_w(n+1) = C_w(n+1) - C_w(n) + 2$ for all $n$, $0 \leq n \leq |w|$. 
More Stuff on Rich Words

G.-Justin-Widmer-Zamboni, Palindromic richness, 2008

- *almost rich words*: a new palindrome is introduced at all, but a finite number of positions
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Example: \((pq)\omega = pqpqpq\cdots\) where \(p, q\) are palindromes
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  Example: \((aacbccbcacbc)^\omega = aacbccbcacbc aacbccbcacbc \cdots\)
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**Open Problems**

- Characterisation of morphisms that preserve (almost) richness
- Enumeration of rich words
Thank You!

Dammit, I’m mad!

U R 2 R U?

* Both phrases are (rich) palindromes! *