Outline

1. Background
   - Repetitions & patterns in words
   - Crucial words & abelian powers

2. Minimal crucial words avoiding abelian cubes
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   - Lower bound for length

3. Minimal crucial words avoiding abelian $k$-th powers
   - Upper bound for length

4. Further research
1 Background
   - Repetitions & patterns in words
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   - Upper bound for length
   - Lower bound for length

3 Minimal crucial words avoiding abelian $k$-th powers
   - Upper bound for length

4 Further research
A \textit{word} \( w \) is a finite or infinite sequence of symbols (\textit{letters}) taken from a non-empty finite set \( A \) (\textit{alphabet}).

Example with \( A = \{a, b, c\} \):

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w = abca, \quad w^\infty = (abca)^\infty = abcaabcaabcaabca \cdots .
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Example: $|abca| = 4$. 

Amy Glen  (Reykjavík University) 
Crucial words for abelian repetitions 
June 2009 4 / 26
Repetitions in words

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- A factor of a word $w$ is a block of consecutive letters in $w$.

  Example: $w = abca$ has 9 distinct factors

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- **Fact:** Over a 2-letter alphabet $\{a, b\}$, any word $w$ with $|w| > 3$ must have a factor of the form $XX = X^2$, called a square.
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Amy Glen (Reykjavík University)  
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  \]

  gives (in the limit) the infinite word

  \[
  cbacabcbabcabcabcabca \ldots
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Repetitions in words . . .

- **Thue (1912):** also constructed an infinite word over \( \{a, b\} \) avoiding factors of the form

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XXX = X^3 \text{ (called } \textit{cubes}) \quad \text{and} \quad XYXYX \text{ (called } \textit{overlaps}).
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- Now called the *Thue-Morse word* as it was rediscovered by Morse in 1921 (in the context of symbolic dynamics).
Pattern avoidance

- Patterns such as $X$, $XYX$, $XYZXYX$ (called *sesquipowers*) cannot be avoided by infinite words (i.e., they are *unavoidable*).
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  Connections to *semigroup theory*, *formal language theory*, *symbolic dynamics*, ...
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- Erdős (1961): introduced a commutative version of Thue’s problem.

  *Does there exist an infinite word over a fixed finite alphabet containing no abelian squares, i.e., avoiding factors of the form $XX'$ where $X'$ is a permutation of $X$?*
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- Answer: *YES.*
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- **Answer:** YES. Existence was established for alphabets of size:
  
  - 25 and improved to 7 (A. Evdokimov, 1968 & 1971);
  - 5 (P.A.B. Pleasants, 1970);
  - 4 (Keränen, 1992), the *optimal result* (such a word does not exist over a 3-letter alphabet).
Abelian powers

Let $\mathcal{A}_n = \{1, 2, \ldots, n\}$ and let $k \geq 2$ be an integer.
Abelian powers

Let $A_n = \{1, 2, \ldots, n\}$ and let $k \geq 2$ be an integer.

- A word $W$ over $A_n$ contains a $k$-th power if $W$ has a factor of the form $X^k = XX \ldots X$ ($k$ times) for some non-empty word $X$. 
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- Example:
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  V = 13243232323243 \text{ contains the 4-th power } (32)^4 = 32323232.
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- A word $W$ contains an \textit{abelian k-th power} if $W$ has a factor of the form $X_1X_2 \ldots X_k$ where $X_i$ is a permutation of $X_1$ for $2 \leq i \leq k$. 
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- The cases \( k = 2 \) and \( k = 3 \) give us (abelian) \textit{squares} and \textit{cubes}.
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  The cases $k = 2$ and $k = 3$ give us (abelian) squares and cubes.

  Examples:

  - $V$ contains the abelian square $43232\, 32324$.
  - $123\, 312\, 213$ is an abelian cube.
Abelian powers . . .

- A word is \textit{(abelian) $k$-power-free} if it \textit{avoids} (abelian) $k$-th powers.

  \textbf{Example:} 1234324 is abelian cube-free, but \textbf{not} abelian square-free since it contains the abelian square $234\ 324$. 

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Note: “Does there exist an infinite abelian $k$-power-free word on $n$ letters?” is equivalent to asking

“Does there exist an infinite semigroup $S$ having $n$ generators and satisfying the property that any abelian $k$-th power in $S$ vanishes to the identity element?”
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For instance, the answer is known to be YES for:

- \( k = 2 \) and \( n = 4 \) (Keränen, 1992)
- \( k = 3 \) and \( n = 3 \) (Dekking, 1979)
- \( k = 4 \) and \( n = 2 \) (Dekking, 1979)
- \( k = 5 \) and \( n = 2 \) (Justin, 1972)
In algebraic contexts, it is also natural to consider words avoiding other patterns too . . .
Pattern avoidance & semigroup theory

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For instance:

In the study of semigroup varieties, the following Burnside type question is natural:

"Given an arbitrary pattern $p$ in one or more variables, does there exist a finitely generated infinite semigroup satisfying the law $p = e$, where $e$ is the identity element?"
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[See Currie-Linek (2001) for more details and references.]

We are interested in a particular problem in relation to words avoiding abelian powers . . .
A word $W$ over $A_n$ is crucial with respect to a given set of prohibited words (or simply prohibitions) if $W$ avoids the prohibitions, but $Wx$ does not avoid the prohibitions for any letter $x$ occurring in $W$. 
Crucial words with respect to abelian powers

- A word $W$ over $\mathcal{A}_n$ is crucial with respect to a given set of prohibited words (or simply prohibitions) if $W$ avoids the prohibitions, but $Wx$ does not avoid the prohibitions for any letter $x$ occurring in $W$.

A minimal crucial word is a crucial word of the shortest length.
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Example: $W = 21211$ is crucial with respect to abelian cubes since:
A word $W$ over $A_n$ is \textit{crucial} with respect to a given set of \textit{prohibited words} (or simply \textit{prohibitions}) if $W$ avoids the prohibitions, but $Wx$ does not avoid the prohibitions for any letter $x$ occurring in $W$.

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Example: $W = 21211$ is crucial with respect to abelian cubes since:

- $W$ is abelian cube-free;
- $W1$ and $W2$ end with the abelian cubes $111$ and $212112$, respectively.
A word $W$ over $A_n$ is crucial with respect to a given set of prohibited words (or simply prohibitions) if $W$ avoids the prohibitions, but $Wx$ does not avoid the prohibitions for any letter $x$ occurring in $W$. 

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Example: $W = 21211$ is crucial with respect to abelian cubes since:

- $W$ is abelian cube-free;
- $W1$ and $W2$ end with the abelian cubes 111 and 212112, respectively.

In fact, $W$ is a minimal crucial word over $\{1, 2\}$ with respect to abelian cubes.
Zimin words

- Problems of the type proposed by Erdős in 1961 were also considered by Zimin (1984) in the non-abelian sense.
Zimin words

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- The **Zimin word** $Z_n$ over $A_n$ is defined recursively as follows:

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Z_1 = 1 \quad \text{and} \quad Z_n = Z_{n-1} n Z_{n-1} \quad \text{for } n \geq 2.
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- $Z_2 = 121$
- $Z_3 = 1213121$
- $Z_4 = 121312141213121$
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  The first four Zimin words are:

  $Z_1 = 1$
  $Z_2 = 121$
  $Z_3 = 1213121$
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- The *$k$-generalised Zimin word* $Z_{n,k} = X_n$ is defined as

  \[ X_1 = 1^{k-1} = 11\ldots1, \quad X_n = (X_{n-1}n)^{k-1}X_{n-1} = X_{n-1}nX_{n-1}n\ldots nX_{n-1} \]

  where the number of 1’s, as well as the number of $n$’s, is $k – 1$. 
Zimin words . . .

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\[ Z_{1,3} = 11 \]
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  \[
  Z_{1,3} = 11 \\
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Zimin words . . .

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Note:
- \( Z_n = Z_{n,2} \).
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Note:

- \( Z_n = Z_{n,2} \).
- \( Z_{n,k} \) is crucial with respect to abelian \( k \)-th powers.
The first three 3-generalised Zimin words are:

\[ Z_{1,3} = 11 \]
\[ Z_{2,3} = 11211211 \]
\[ Z_{3,3} = 112112111311211211311211211 \]

Note:

- \( Z_n = Z_{n,2} \).
- \( Z_{n,k} \) is crucial with respect to abelian \( k \)-th powers.
- \( Z_{n,k} \) has length \( k^n - 1 \).
Zimin words . . .

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Much less is known in the case of abelian \( k \)-th powers...
Minimal crucial words avoiding abelian powers

Minimal crucial words avoiding abelian powers


- Evdokimov-Kitaev (2004): proved that a minimal crucial abelian square-free word over an $n$-letter alphabet has length $4n - 7$ for $n \geq 3$. 

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Now we extend the study of crucial abelian $k$-power-free words to the case of $k > 2$.

- We provide a complete solution to the problem of determining the length of a minimal crucial abelian cube-free word (the case $k = 3$).
- And we conjecture a solution in the general case.
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  - We provide a complete solution to the problem of determining the length of a minimal crucial abelian cube-free word (the case $k = 3$).
  - And we conjecture a solution in the general case.

- Let $\ell_k(n)$ denote the length of a minimal crucial word over $A_n$ avoiding abelian $k$-th powers.
Outline

1 Background
   - Repetitions & patterns in words
   - Crucial words & abelian powers

2 Minimal crucial words avoiding abelian cubes
   - Upper bound for length
   - Lower bound for length

3 Minimal crucial words avoiding abelian $k$-th powers
   - Upper bound for length

4 Further research
Upper bound for $\ell_3(n)$

Note: $Z_{n,3}$ crucial with respect to abelian cubes $\implies \ell_3(n) \leq 3^n - 1.$
Upper bound for $\ell_3(n)$

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Upper bound for $\ell_3(n)$

Note: $Z_{n,3}$ crucial with respect to abelian cubes $\implies \ell_3(n) \leq 3^n - 1$.

- Let $X$ be a crucial word over $A_n$ with respect to abelian $k$-th powers.
- If $X$ is minimal, we may assume w.l.o.g. that $X^n$ is an abelian $k$-th power, and we write:

$$X = \Omega_{n,1} \Omega_{n,2} \cdots \Omega_{n,k}$$

where $\Omega_{n,k} = \Omega'_{n,k} n$

and the $k$ blocks $\Omega_{n,j}$ are permutations of one another.
Upper bound for $\ell_3(n)$

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A construction of crucial abelian cube-free words over $A_n$ for $n \geq 4$: 
Upper bound for $\ell_3(n)$

Note: $Z_{n,3}$ crucial with respect to abelian cubes $\implies \ell_3(n) \leq 3^n - 1.$

- Let $X$ be a crucial word over $A_n$ with respect to abelian $k$-th powers.
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$$X = \Omega_{n,1}\Omega_{n,2} \cdots \Omega'_{n,k}$$

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and the $k$ blocks $\Omega_{n,j}$ are permutations of one another.

A construction of crucial abelian cube-free words over $A_n$ for $n \geq 4$:

**Basis:** Minimal crucial abelian square-free words $W_n = W_{n,2}$ given by Evdokimov \& Kitaev (2004). For $n = 4, 5, 6$:

$W_{4,2} = 34231 \ 3231,$
$W_{5,2} = 4534231 \ 432341,$
$W_{6,2} = 564534231 \ 54323451,$

where spaces separate the blocks.
Minimal crucial abelian square-free words

General construction of $W_{n,2} = \Omega_{n,1} \Omega'_{n,2}$ for $n \geq 4$:

- 1st block $\Omega_{n,1}$: adjoin the factors $i(i+1)$ for $i = n-1, n-2, \ldots, 2$, followed by the letter 1.
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  **Example**: For $n = 4$, we have $\Omega_{4,1} = 34231$. 
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General construction of $W_{n,2} = \Omega_{n,1}\Omega'_{n,2}$ for $n \geq 4$:

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2. **2nd block $\Omega'_{n,2}$**: adjoin the factors $(n - 1)(n - 2) \ldots 432$, then $34 \ldots (n - 2)(n - 1)$, and finally the letter 1.
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   **Example**: For $n = 4$, we have $\Omega'_{4,2} = 3231$. 
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General construction of \( W_{n,2} = \Omega_{n,1} \Omega'_{n,2} \) for \( n \geq 4 \):

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For \( n = 4, 5, 6, 7 \), we have:

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Minimal crucial abelian square-free words

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For $n = 4, 5, 6, 7$, we have:

$W_{4,2} = 34$
Minimal crucial abelian square-free words

General construction of $W_{n,2} = \Omega_{n,1}\Omega'_{n,2}$ for $n \geq 4$:

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- **2nd block $\Omega'_{n,2}$**: adjoin the factors $(n-1)(n-2)\ldots432$, then $34\ldots(n-2)(n-1)$, and finally the letter 1.
  
  **Example**: For $n = 4$, we have $\Omega_{4,2} = 3231$.

For $n = 4, 5, 6, 7$, we have:

$W_{4,2} = 3423$
Minimal crucial abelian square-free words

General construction of $W_{n,2} = \Omega_{n,1}\Omega'_{n,2}$ for $n \geq 4$: 

- **1st block $\Omega_{n,1}$**: adjoin the factors $i(i + 1)$ for $i = n - 1, n - 2, \ldots, 2$, followed by the letter 1.
  
  **Example**: For $n = 4$, we have $\Omega_{4,1} = 34231$.

- **2nd block $\Omega'_{n,2}$**: adjoin the factors $(n - 1)(n - 2)\ldots432$, then $34\ldots(n - 2)(n - 1)$, and finally the letter 1.
  
  **Example**: For $n = 4$, we have $\Omega_{4,2} = 3231$.

For $n = 4, 5, 6, 7$, we have:

$W_{4,2} = 34231$
Minimal crucial abelian square-free words

General construction of $W_{n,2} = \Omega_{n,1}\Omega'_{n,2}$ for $n \geq 4$:

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  **Example**: For $n = 4$, we have $\Omega_{4,2} = 3231$.

For $n = 4, 5, 6, 7$, we have:

$W_{4,2} = 34231 32$
Minimal crucial abelian square-free words

General construction of $W_{n,2} = \Omega_{n,1}\Omega'_{n,2}$ for $n \geq 4$:

- **1st block $\Omega_{n,1}$**: adjoin the factors $i(i+1)$ for $i = n-1, n-2, \ldots, 2$, followed by the letter 1.
  
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- **2nd block $\Omega'_{n,2}$**: adjoin the factors $(n-1)(n-2)\ldots432$, then $34\ldots(n-2)(n-1)$, and finally the letter 1.
  
  **Example**: For $n = 4$, we have $\Omega_{4,2} = 3231$.

For $n = 4, 5, 6, 7$, we have:

$W_{4,2} = 34231 \ 323$
Minimal crucial abelian square-free words

General construction of $W_{n,2} = \Omega_{n,1}\Omega'_{n,2}$ for $n \geq 4$:

- **1st block $\Omega_{n,1}$**: adjoin the factors $i(i+1)$ for $i = n-1, n-2, \ldots, 2$, followed by the letter 1.

  **Example**: For $n = 4$, we have $\Omega_{4,1} = 34231$.

- **2nd block $\Omega'_{n,2}$**: adjoin the factors $(n-1)(n-2)\ldots432$, then $34\ldots(n-2)(n-1)$, and finally the letter 1.

  **Example**: For $n = 4$, we have $\Omega_{4,2} = 3231$.

For $n = 4, 5, 6, 7$, we have:

$W_{4,2} = 34231\ 3231$
Minimal crucial abelian square-free words

General construction of \( W_{n,2} = \Omega_{n,1} \Omega'_{n,2} \) for \( n \geq 4 \):

- **1st block \( \Omega_{n,1} \):** adjoin the factors \( i(i + 1) \) for \( i = n - 1, n - 2, \ldots, 2 \), followed by the letter 1.

  **Example:** For \( n = 4 \), we have \( \Omega_{4,1} = 34231 \).

- **2nd block \( \Omega'_{n,2} \):** adjoin the factors \( (n - 1)(n - 2) \ldots 432 \), then \( 34 \ldots (n - 2)(n - 1) \), and finally the letter 1.

  **Example:** For \( n = 4 \), we have \( \Omega_{4,2} = 3231 \).

For \( n = 4, 5, 6, 7 \), we have:

\[
W_{4,2} = 34231 \ 3231 \\
W_{5,2} =
\]
Minimal crucial abelian square-free words

General construction of $W_{n,2} = \Omega_{n,1}\Omega'_{n,2}$ for $n \geq 4$:

- **1st block $\Omega_{n,1}$**: adjoin the factors $i(i+1)$ for $i = n-1, n-2, \ldots, 2$, followed by the letter 1.

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  **Example**: For $n = 4$, we have $\Omega_{4,2} = 3231$.  

For $n = 4, 5, 6, 7$, we have:

$W_{4,2} = 34231$ $3231$

$W_{5,2} = 45$
Minimal crucial abelian square-free words

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  **Example:** For \( n = 4 \), we have \( \Omega_{4,1} = 34231 \).

- **2nd block** \( \Omega'_{n,2} \): adjoin the factors \((n-1)(n-2)\ldots432\), then \( 34\ldots(n-2)(n-1) \), and finally the letter 1.
  
  **Example:** For \( n = 4 \), we have \( \Omega_{4,2} = 3231 \).

For \( n = 4, 5, 6, 7 \), we have:

\[
W_{4,2} = 34231 \ 3231
\]
\[
W_{5,2} = 4534
\]
Minimal crucial abelian square-free words

General construction of \( W_{n,2} = \Omega_{n,1} \Omega'_{n,2} \) for \( n \geq 4 \):

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  Example: For \( n = 4 \), we have \( \Omega_{4,1} = 34231 \).

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  Example: For \( n = 4 \), we have \( \Omega_{4,2} = 3231 \).

For \( n = 4, 5, 6, 7 \), we have:

\[ W_{4,2} = 34231 \ 3231 \]

\[ W_{5,2} = 453423 \]
Minimal crucial abelian square-free words

General construction of $W_{n,2} = \Omega_{n,1}\Omega'_{n,2}$ for $n \geq 4$:

- **1st block $\Omega_{n,1}$:** adjoin the factors $i(i + 1)$ for $i = n - 1, n - 2, \ldots, 2$, followed by the letter 1.
  
  **Example:** For $n = 4$, we have $\Omega_{4,1} = 34231$.

- **2nd block $\Omega'_{n,2}$:** adjoin the factors $(n - 1)(n - 2)\ldots432$, then $34\ldots(n - 2)(n - 1)$, and finally the letter 1.
  
  **Example:** For $n = 4$, we have $\Omega_{4,2} = 3231$.

For $n = 4, 5, 6, 7$, we have:

$W_{4,2} = 34231\ 3231$

$W_{5,2} = 4534231$
Minimal crucial abelian square-free words

General construction of $W_{n,2} = \Omega_{n,1}\Omega'_{n,2}$ for $n \geq 4$:

- **1st block $\Omega_{n,1}$**: Adjoin the factors $i(i+1)$ for $i = n-1, n-2, \ldots, 2$, followed by the letter 1.
  
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- **2nd block $\Omega'_{n,2}$**: Adjoin the factors $(n-1)(n-2)\ldots432$, then $34\ldots(n-2)(n-1)$, and finally the letter 1.
  
  **Example**: For $n = 4$, we have $\Omega'_{4,2} = 3231$.

For $n = 4, 5, 6, 7$, we have:

- $W_{4,2} = 34231\ 3231$
- $W_{5,2} = 4534231\ 432$
Minimal crucial abelian square-free words

General construction of $W_{n,2} = \Omega_{n,1}\Omega'_{n,2}$ for $n \geq 4$:

- **1st block $\Omega_{n,1}$**: adjoin the factors $i(i + 1)$ for $i = n - 1, n - 2, \ldots, 2$, followed by the letter $1$.

  **Example**: For $n = 4$, we have $\Omega_{4,1} = 34231$.

- **2nd block $\Omega'_{n,2}$**: adjoin the factors $(n - 1)(n - 2)\ldots 432$, then $34\ldots(n - 2)(n - 1)$, and finally the letter $1$.

  **Example**: For $n = 4$, we have $\Omega_{4,2} = 3231$.

For $n = 4, 5, 6, 7$, we have:

$W_{4,2} = 34231 \ 3231$

$W_{5,2} = 4534231 \ 43234$
Minimal crucial abelian square-free words

General construction of $W_{n,2} = \Omega_{n,1}\Omega'_{n,2}$ for $n \geq 4$:

- **1st block $\Omega_{n,1}$**: adjoin the factors $i(i+1)$ for $i = n-1, n-2, \ldots, 2$, followed by the letter 1.
  
  **Example**: For $n = 4$, we have $\Omega_{4,1} = 34231$.

- **2nd block $\Omega'_{n,2}$**: adjoin the factors $(n-1)(n-2)\ldots432$, then $34\ldots(n-2)(n-1)$, and finally the letter 1.
  
  **Example**: For $n = 4$, we have $\Omega_{4,2} = 3231$.

For $n = 4, 5, 6, 7$, we have:

$W_{4,2} = 34231\ 3231$

$W_{5,2} = 4534231\ 432341$
Minimal crucial abelian square-free words

General construction of \( W_{n,2} = \Omega_{n,1} \Omega'_{n,2} \) for \( n \geq 4 \):

- **1st block \( \Omega_{n,1} \):** adjoin the factors \( i(i + 1) \) for \( i = n - 1, n - 2, \ldots, 2 \), followed by the letter 1.
  
  **Example:** For \( n = 4 \), we have \( \Omega_{4,1} = 34231 \).

- **2nd block \( \Omega'_{n,2} \):** adjoin the factors \( (n - 1)(n - 2) \ldots 432 \), then \( 34 \ldots (n - 2)(n - 1) \), and finally the letter 1.

  **Example:** For \( n = 4 \), we have \( \Omega_{4,2} = 3231 \).

For \( n = 4, 5, 6, 7 \), we have:

\[
\begin{align*}
W_{4,2} &= 34231 \ 3231 \\
W_{5,2} &= 4534231 \ 432341 \\
W_{6,2} &= 564534231 \ 54323451 \\
W_{7,2} &= 67564534231 \ 6543234561
\end{align*}
\]
An optimal construction

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1} \Omega_{n,2} \Omega'_{n,3}$:
An optimal construction

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$...

Construction of $W_{n,3} = \Omega_{n,1} \Omega_{n,2} \Omega'_{n,3}$:

$W_{4,2} = 34231 \ 3231 \rightarrow$
An optimal construction

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1} \Omega_{n,2} \Omega'_{n,3}$:

$W_{4,2} = 34231 \ 3231 \rightarrow \underline{34231} \ 34231 \ 3231$

Duplicate 1st block
An optimal construction

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \rightarrow 34231\ 34231\underline{134}\ 3231$

Append 134 to 2nd block
An optimal construction

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1} \Omega_{n,2} \Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \rightarrow 344231\ 34231134\ 3231$

Duplicate rightmost $x$ for each $x \neq 2$ in 1st & 3rd blocks
An optimal construction

For \( n \geq 4 \), we obtain crucial abelian cube-free words \( W_{n,3} \) from \( W_{n,2} \) . . .

Construction of \( W_{n,3} = \Omega_{n,1} \Omega_{n,2} \Omega'_{n,3} \):

\( W_{4,2} = 34231 \; 3231 \quad \rightarrow \quad 34423\color{red}{31} \; 34231134 \; 3231 \)

Duplicate rightmost \( x \) for each \( x \neq 2 \) in 1st & 3rd blocks
An optimal construction

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \rightarrow \ 3442331\underline{1}\ 34231134\ 3231$

Duplicate rightmost $x$ for each $x \neq 2$ in 1st & 3rd blocks
An optimal construction

For \( n \geq 4 \), we obtain crucial abelian cube-free words \( W_{n,3} \) from \( W_{n,2} \) . . .

Construction of \( W_{n,3} = \Omega_{n,1} \Omega_{n,2} \Omega'_{n,3} \):

\[
W_{4,2} = 34231 \ 3231 \longrightarrow \ 34423111 \ 34231134 \ 32331
\]

Duplicate rightmost \( x \) for each \( x \neq 2 \) in 1st & 3rd blocks
An optimal construction

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2} \ldots$

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231 \ 3231 \rightarrow \ 34423311 \ 34231134 \ 323311$

Duplicate rightmost $x$ for each $x \neq 2$ in 1st & 3rd blocks
An optimal construction

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231 \ 3231 \rightarrow \ 34423311 \ 34231134 \ 3233411$

Insert 4 before leftmost 1 in 3rd block
An optimal construction

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1} \Omega_{n,2} \Omega'_{n,3}$:

$W_{4,2} = 34231 3231 \longrightarrow 34423311 34231134 3233411 = W_{4,3}$
An optimal construction

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$.

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

\[
W_{4,2} = 34231 \ 3231 \quad \rightarrow \quad 34423311 \ 34231134 \ 3233411 = W_{4,3}
\]

\[
W_{5,2} = 4534231 \ 432341 \quad \rightarrow
\]
An optimal construction

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1} \Omega_{n,2} \Omega'_{n,3}$:

$W_{4,2} = 34231 \ 3231 \rightarrow 34423311 \ 34231134 \ 3233411 = W_{4,3}$

$W_{5,2} = 4534231 \ 432341$

$\rightarrow 4534231 \ 4534231 \ 432341$

Duplicate 1st block
An optimal construction

For \( n \geq 4 \), we obtain crucial abelian cube-free words \( W_{n,3} \) from \( W_{n,2} \) . . .

Construction of \( W_{n,3} = \Omega_{n,1} \Omega_{n,2} \Omega'_{n,3} \):

\[
W_{4,2} = 34231 \ 3231 \rightarrow 34423311 \ 34231134 \ 3233411 = W_{4,3}
\]

\[
W_{5,2} = 4534231 \ 432341
\]

\[
\rightarrow 4534231 \ 4534231 \ 1345 \ 432341
\]

Append 1345 to 2nd block
An optimal construction

For \( n \geq 4 \), we obtain crucial abelian cube-free words \( W_{n,3} \) from \( W_{n,2} \) . . .

Construction of \( W_{n,3} = \Omega_{n,1} \Omega_{n,2} \Omega'_{n,3} \):

\[
W_{4,2} = 34231\ 3231 \rightarrow 34423311\ 34231134\ 3233411 = W_{4,3}
\]

\[
W_{5,2} = 4534231\ 432341 \\
\rightarrow 45\underline{5}\ 34231\ 45342311345\ 432341
\]

Duplicate rightmost \( x \) for each \( x \neq 2 \) in 1st & 3rd blocks
An optimal construction

For \( n \geq 4 \), we obtain crucial abelian cube-free words \( W_{n,3} \) from \( W_{n,2} \) . . .

Construction of \( W_{n,3} = \Omega_{n,1} \Omega_{n,2} \Omega'_{n,3} \):

\[
W_{4,2} = 342313231 \longrightarrow 3442331134231341134 = W_{4,3}
\]

\[
W_{5,2} = 4534231432341 \longrightarrow 45534423145342311345432341
\]

Duplicate rightmost \( x \) for each \( x \neq 2 \) in 1st & 3rd blocks
An optimal construction

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2} \ldots$

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231 \ 3231 \rightarrow 34423311 \ 34231134 \ 3233411 = W_{4,3}$

$W_{5,2} = 4534231 \ 432341$

$\rightarrow 455344231 \ 45342311345 \ 432341$

Duplicate rightmost $x$ for each $x \neq 2$ in 1st & 3rd blocks
An optimal construction

For \( n \geq 4 \), we obtain crucial abelian cube-free words \( W_{n,3} \) from \( W_{n,2} \) . . .

Construction of \( W_{n,3} = \Omega_{n,1} \Omega_{n,2} \Omega'_{n,3} \):

\[
W_{4,2} = 34231 \ 3231 \quad \rightarrow \quad 34423311 \ 34231134 \ 3233411 = W_{4,3}
\]

\[
W_{5,2} = 4534231 \ 432341
\]

\[
\rightarrow 4553442331 \ 1 \ 45342311345 \ 432341
\]

Duplicate rightmost \( x \) for each \( x \neq 2 \) in 1st & 3rd blocks
An optimal construction

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1} \Omega_{n,2} \Omega'_{n,3}$:

$W_{4,2} = 34231 \ 3231 \rightarrow \ 34423311 \ 34231134 \ 3233411 = W_{4,3}$

$W_{5,2} = 4534231 \ 432341$

$\rightarrow 45534423311 \ 45342311345 \ 4323341$

Duplicate rightmost $x$ for each $x \neq 2$ in 1st & 3rd blocks
An optimal construction

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$. . .

Construction of $W_{n,3} = \Omega_{n,1} \Omega_{n,2} \Omega_{n,3}'$:

$W_{4,2} = 34231\ 3231 \rightarrow 34423311\ 34231134\ 3233411 = W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\rightarrow 45534423311\ 45342311345\ 4323341$  

Duplicate rightmost $x$ for each $x \neq 2$ in 1st & 3rd blocks
An optimal construction

For \( n \geq 4 \), we obtain crucial abelian cube-free words \( W_{n,3} \) from \( W_{n,2} \) . . .

Construction of \( W_{n,3} = \Omega_{n,1} \Omega_{n,2} \Omega'_{n,3} \):

\[
W_{4,2} = 34231 \ 3231 \quad \rightarrow \quad 34423311 \ 34231134 \ 3233411 = W_{4,3}
\]

\[
W_{5,2} = 4534231 \ 432341
\]

\[
\rightarrow \quad 45534423311 \ 45342311345 \ 43233441 \underline{1}
\]

Duplicate rightmost \( x \) for each \( x \neq 2 \) in 1st & 3rd blocks
An optimal construction

For \( n \geq 4 \), we obtain crucial abelian cube-free words \( W_{n,3} \) from \( W_{n,2} \ldots \)

Construction of \( W_{n,3} = \Omega_{n,1} \Omega_{n,2} \Omega'_{n,3} \):  

\[
W_{4,2} = 34231 \ 3231 \quad \rightarrow \quad 34423311 \ 34231134 \ 3233411 = W_{4,3}
\]

\[
W_{5,2} = 4534231 \ 432341
\]

\[
\rightarrow \ 45534423311 \ 45342311345 \ 4323344511
\]

Insert 5 before leftmost 1 in 3rd block
An optimal construction

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1} \Omega_{n,2} \Omega'_{n,3}$:

$W_{4,2} = 34231 \ 3231 \longrightarrow 34423311 \ 34231134 \ 3233411 = W_{4,3}$

$W_{5,2} = 4534231 \ 432341$

$\longrightarrow 45534423311 \ 45342311345 \ 4323344511 = W_{5,3}$
An optimal construction

For \( n \geq 4 \), we obtain crucial abelian cube-free words \( W_{n,3} \) from \( W_{n,2} \) . . .

Construction of \( W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3} \):

\[
W_{4,2} = 34231\ 3231 \quad \rightarrow \quad 34423311\ 34231134\ 3233411 = W_{4,3}
\]

\[
W_{5,2} = 4534231\ 432341
\]

\( \rightarrow \quad 45534423311\ 45342311345\ 4323344511 = W_{5,3} \)

\[
W_{6,2} = 564534231\ 54323451
\]

\( \rightarrow \)
An optimal construction

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\,3231 \rightarrow 34423311\,34231134\,3233411 = W_{4,3}$

$W_{5,2} = 4534231\,432341 \rightarrow 45534423311\,45342311345\,4323344511 = W_{5,3}$

$W_{6,2} = 564534231\,54323451 \rightarrow 564534231\,564534231\,54323451$

Duplicate 1st block
An optimal construction

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\, 3231 \quad \rightarrow \quad 34423311\, 34231134\, 3233411 = W_{4,3}$

$W_{5,2} = 4534231\, 432341$

$\quad \rightarrow \quad 45534423311\, 45342311345\, 4323344511 = W_{5,3}$

$W_{6,2} = 564534231\, 54323451$

$\quad \rightarrow \quad 564534231\, 564534231\, 13456\, 54323451$

Append $13456$ to 2nd block
An optimal construction

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231 \ 3231 \rightarrow 34423311 \ 34231134 \ 3233411 = W_{4,3}$

$W_{5,2} = 4534231 \ 432341$

$\rightarrow 45534423311 \ 45342311345 \ 4323344511 = W_{5,3}$

$W_{6,2} = 564534231 \ 54323451$

$\rightarrow 566455344423311 \ 56453423113456 \ 543233445511$

Duplicate rightmost $x$ for each $x \neq 2$ in 1st & 3rd blocks
An optimal construction

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1} \Omega_{n,2} \Omega'_{n,3}$:

$W_{4,2} = 34231 \ 3231 \rightarrow 34423311 \ 34231134 \ 3233411 = W_{4,3}$

$W_{5,2} = 4534231 \ 432341$

$\rightarrow 45534423311 \ 45342311345 \ 4323344511 = W_{5,3}$

$W_{6,2} = 564534231 \ 54323451$

$\rightarrow 56645534423311 \ 56453423113456 \ 5432334455611$

Insert 6 before leftmost 1 in 3rd block
An optimal construction

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \rightarrow 34423311\ 34231134\ 3233411 = W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\rightarrow 45534423311\ 45342311345\ 4323344511 = W_{5,3}$

$W_{6,2} = 564534231\ 54323451$

$\rightarrow 56645534423311\ 56453423113456\ 5432334455611 = W_{6,3}$
An optimal construction

For \( n \geq 4 \), we obtain crucial abelian cube-free words \( W_{n,3} \) from \( W_{n,2} \) . . .

Construction of \( W_{n,3} = \Omega_{n,1} \Omega_{n,2} \Omega'_{n,3} \):

\[
W_{4,2} = 34231 \ 3231 \quad \rightarrow \quad 34423311 \ 34231134 \ 3233411 = W_{4,3}
\]

\[
W_{5,2} = 4534231 \ 432341
\]

\[
\rightarrow \quad 45534423311 \ 45342311345 \ 4323344511 = W_{5,3}
\]

\[
W_{6,2} = 564534231 \ 54323451
\]

\[
\rightarrow \quad 56645534423311 \ 56453423113456 \ 5432334455611 = W_{6,3}
\]

\[
W_{7,2} = 67564534231 \ 6543234561
\]

\[
\rightarrow
\]
An optimal construction

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \rightarrow 34423311\ 34231134\ 3233411 = W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\rightarrow 45534423311\ 45342311345\ 4323344511 = W_{5,3}$

$W_{6,2} = 564534231\ 54323451$

$\rightarrow 56645534423311\ 56453423113456\ 5432334455611 = W_{6,3}$

$W_{7,2} = 67564534231\ 6543234561$

$\rightarrow 67564534231\ 67564534231\ 6543234561$
An optimal construction

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1} \Omega_{n,2} \Omega'_{n,3}$:

$W_{4,2} = 34231 \ 3231 \rightarrow 34423311 \ 34231134 \ 3233411 = W_{4,3}$

$W_{5,2} = 4534231 \ 432341$

$\rightarrow 45534423311 \ 45342311345 \ 4323344511 = W_{5,3}$

$W_{6,2} = 564534231 \ 54323451$

$\rightarrow 56645534423311 \ 56453423113456 \ 5432334455611 = W_{6,3}$

$W_{7,2} = 67564534231 \ 6543234561$

$\rightarrow 67564534231 \ 67564534231 \textbf{134567} \ 6543234561$
An optimal construction

For \( n \geq 4 \), we obtain crucial abelian cube-free words \( W_{n,3} \) from \( W_{n,2} \) ...

Construction of \( W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3} \):

\[
\begin{align*}
W_{4,2} &= 34231\ 3231 \quad \rightarrow \quad 34423311\ 34231134\ 3233411 = W_{4,3} \\
W_{5,2} &= 4534231\ 432341 \\
&\quad \rightarrow \quad 45534423311\ 45342311345\ 4323344511 = W_{5,3} \\
W_{6,2} &= 564534231\ 54323451 \\
&\quad \rightarrow \quad 56645534423311\ 56453423113456\ 5432334455611 = W_{6,3} \\
W_{7,2} &= 67564534231\ 6543234561 \\
&\quad \rightarrow \quad 67756645534442331\ 67564534231134567\ 65432334455611
\end{align*}
\]
An optimal construction

For \( n \geq 4 \), we obtain crucial abelian cube-free words \( W_{n,3} \) from \( W_{n,2} \) . . .

Construction of \( W_{n,3} = \Omega_{n,1} \Omega_{n,2} \Omega'_{n,3} \):

\[
\begin{align*}
W_{4,2} &= 34231\ 3231 \\
&\quad\rightarrow 34423311\ 34231134\ 3233411 = W_{4,3} \\
W_{5,2} &= 4534231\ 432341 \\
&\quad\rightarrow 45534423311\ 45342311345\ 4323344511 = W_{5,3} \\
W_{6,2} &= 564534231\ 54323451 \\
&\quad\rightarrow 56645534423311\ 56453423113456\ 5432334455611 = W_{6,3} \\
W_{7,2} &= 67564534231\ 6543234561 \\
&\quad\rightarrow 67756645534423311\ 67564534231134567\ 6543233445566711
\end{align*}
\]
An optimal construction

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2} \ldots$

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231 \ 3231 \rightarrow 34423311 \ 34231134 \ 3233411 = W_{4,3}$

$W_{5,2} = 4534231 \ 432341$  
$\rightarrow 45534423311 \ 45342311345 \ 4323344511 = W_{5,3}$

$W_{6,2} = 564534231 \ 54323451$  
$\rightarrow 56645534423311 \ 56453423113456 \ 5432334455611 = W_{6,3}$

$W_{7,2} = 67564534231 \ 6543234561$  
$\rightarrow 67756645534423311 \ 67564534231134567 \ 6543233445566711$
An optimal construction

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1} \Omega_{n,2} \Omega_{n,3}'$:

$W_{4,2} = 34231\ 3231 \rightarrow 34423311\ 34231134\ 32334111 = W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\rightarrow 45534423311\ 45342311345\ 43233445111 = W_{5,3}$

$W_{6,2} = 564534231\ 54323451$

$\rightarrow 56645534423311\ 56453423113456\ 54323344556111 = W_{6,3}$

$W_{7,2} = 67564534231\ 6543234561$

$\rightarrow 67756645534423311\ 67564534231134567\ 6543233445566711$

Note:

$W_{n,3} = (n - 1)n\Omega_{n-1,1} (n - 1)n\Omega_{n-1,2} n(n - 1)\Omega_{n-1,3}'[11]^{-1}(n - 1)n11$. 

Amy Glen (Reykjavík University) 
Crucial words for abelian repetitions 
June 2009
An optimal construction

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231 \ 3231 \rightarrow 34423311 \ 34231134 \ 3233411 = W_{4,3}$

$W_{5,2} = 4534231 \ 432341$

$\rightarrow 45534423311 \ 45342311345 \ 4323344511 = W_{5,3}$

$W_{6,2} = 564534231 \ 54323451$

$\rightarrow 56645534423311 \ 56453423113456 \ 5432334455611 = W_{6,3}$

$W_{7,2} = 67564534231 \ 6543234561$

$\rightarrow 67756645534423311 \ 67564534231134567 \ 6543233445566711$

Note:

$W_{n,3} = (n - 1)n\Omega_{n-1,1} \ (n - 1)n\Omega_{n-1,2}n \ (n - 1)\Omega'_{n-1,3}[11]^{-1} (n - 1)n11$. 
An optimal construction . . .

By construction:

$$W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$$

where $$\Omega_{n,3} = \Omega'_{n,3}n$$ and each $$\Omega_{n,i}$$ contains two 1’s, one 2, two $$n$$’s, and three $$x$$’s for $$x = 3, \ldots, n - 1$$. 
By construction:

\[ W_{n,3} = \Omega_{n,1} \Omega_{n,2} \Omega'_{n,3} \]

where \( \Omega_{n,3} = \Omega'_{n,3} n \) and each \( \Omega_{n,i} \) contains two 1’s, one 2, two \( n \)'s, and three \( x \)'s for \( x = 3, \ldots, n - 1 \).

Hence, \( |W_{n,3}| = 3(3(n - 3) + 2 \cdot 2 + 1) - 1 = 9n - 13 \).
An optimal construction . . .

- By construction:
  \[ W_{n,3} = \Omega_{n,1} \Omega_{n,2} \Omega'_{n,3} \]
  where \( \Omega_{n,3} = \Omega'_{n,3} n \) and each \( \Omega_{n,i} \) contains two 1's, one 2, two \( n \)'s, and three \( x \)'s for \( x = 3, \ldots, n - 1 \).

- Hence, \( |W_{n,3}| = 3(3(n - 3) + 2 \cdot 2 + 1) - 1 = 9n - 13 \).

- Moreover, \( W_{n,3} \) is crucial with respect to abelian cubes.
An optimal construction . . .

- By construction:
  \[ W_{n,3} = \Omega_{n,1} \Omega_{n,2} \Omega'_{n,3} \]
  where \( \Omega_{n,3} = \Omega'_{n,3} n \) and each \( \Omega_{n,i} \) contains two 1's, one 2, two \( n \)'s, and three \( x \)'s for \( x = 3, \ldots, n - 1 \).

- Hence, \(|W_{n,3}| = 3(3(n - 3) + 2 \cdot 2 + 1) - 1 = 9n - 13\).

- Moreover, \( W_{n,3} \) is crucial with respect to abelian cubes.

- Thus, a minimal crucial word avoiding abelian cubes has length at most \( 9n - 13 \) for \( n \geq 4 \).
An optimal construction . . .

- By construction:
  \[ W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3} \]
  where \( \Omega_{n,3} = \Omega'_{n,3} n \) and each \( \Omega_{n,i} \) contains two 1’s, one 2, two \( n \)'s, and three \( x \)'s for \( x = 3, \ldots, n - 1 \).

- Hence, \( |W_{n,3}| = 3(3(n - 3) + 2 \cdot 2 + 1) - 1 = 9n - 13 \).

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- Thus, a minimal crucial word avoiding abelian cubes has length at most \( 9n - 13 \) for \( n \geq 4 \).

**Theorem** (G.-Halldórsson-Kitaev, 2008)

For \( n \geq 4 \), we have \( \ell_3(n) \leq 9n - 13 \).
An optimal construction . . .

- By construction:
  \[ W_{n,3} = \Omega_{n,1} \Omega_{n,2} \Omega'_{n,3} \]
  where \( \Omega_{n,3} = \Omega'_{n,3} n \) and each \( \Omega_{n,i} \) contains two 1's, one 2, two \( n \)'s, and three \( x \)'s for \( x = 3, \ldots, n - 1 \).

- Hence, \( |W_{n,3}| = 3(3(n - 3) + 2 \cdot 2 + 1) - 1 = 9n - 13 \).

- Moreover, \( W_{n,3} \) is crucial with respect to abelian cubes.

- Thus, a minimal crucial word avoiding abelian cubes has length at most \( 9n - 13 \) for \( n \geq 4 \).

Theorem (G.-Halldórsson-Kitaev, 2008)

For \( n \geq 4 \), we have \( \ell_3(n) \leq 9n - 13 \).

This upper bound is optimal . . .
Lower bound for $\ell_3(n)$

By considering the number of possible occurrences of each letter in a crucial word, we use combinatorial arguments to establish the following:

**Theorem** (G.-Halldórsson-Kitaev, 2008)

For $n \geq 5$, we have $\ell_3(n) \geq 9n - 13$. 
Lower bound for $\ell_3(n)$

By considering the number of possible occurrences of each letter in a crucial word, we use combinatorial arguments to establish the following:

**Theorem** (G.-Halldórsson-Kitaev, 2008)

For $n \geq 5$, we have $\ell_3(n) \geq 9n - 13$.

**Corollary** (G.-Halldórsson-Kitaev, 2008)

For $n \geq 5$, we have $\ell_3(n) = 9n - 13$. 
By considering the number of possible occurrences of each letter in a crucial word, we use combinatorial arguments to establish the following:

**Theorem (G.-Halldórsson-Kitaev, 2008)**

For $n \geq 5$, we have $\ell_3(n) \geq 9n - 13$.

**Corollary (G.-Halldórsson-Kitaev, 2008)**

For $n \geq 5$, we have $\ell_3(n) = 9n - 13$.

Note: $\ell_3(n) = 2, 5, 11, 20$ for $n = 1, 2, 3, 4$, respectively.
Lower bound for $\ell_3(n)$

By considering the number of possible occurrences of each letter in a crucial word, we use combinatorial arguments to establish the following:

**Theorem** (G.-Halldórsson-Kitaev, 2008)
For $n \geq 5$, we have $\ell_3(n) \geq 9n - 13$.

**Corollary** (G.-Halldórsson-Kitaev, 2008)
For $n \geq 5$, we have $\ell_3(n) = 9n - 13$.

**Note:** $\ell_3(n) = 2, 5, 11, 20$ for $n = 1, 2, 3, 4$, respectively.

**For example:** 11, 21211, 11231321211, 42131214231211321211.
Outline

1 Background
   • Repetitions & patterns in words
   • Crucial words & abelian powers

2 Minimal crucial words avoiding abelian cubes
   • Upper bound for length
   • Lower bound for length

3 Minimal crucial words avoiding abelian $k$-th powers
   • Upper bound for length

4 Further research
Upper bound for $\ell_k(n)$

Note: $Z_{n,k}$ crucial with respect to abelian $k$-th powers $\implies \ell_k(n) \leq k^n - 1.$
Upper bound for $\ell_k(n)$

Note: $Z_{n,k}$ crucial with respect to abelian $k$-th powers $\implies \ell_k(n) \leq k^n - 1$.

For $n \geq 4$ and $k \geq 2$, we construct a crucial abelian $k$-power-free word $W_{n,k}$ of length $k^2(n - 1) - k - 1$ using the same method as before.
Upper bound for $\ell_k(n)$

Note: $Z_{n,k}$ crucial with respect to abelian $k$-th powers $\Rightarrow \ell_k(n) \leq k^n - 1$.

For $n \geq 4$ and $k \geq 2$, we construct a crucial abelian $k$-power-free word $W_{n,k}$ of length $k^2(n - 1) - k - 1$ using the same method as before.

Theorem (G.-Halldórsson-Kitaev, 2008)

For $n \geq 4$ and $k \geq 2$, we have $\ell_k(n) \leq k^2(n - 1) - k - 1$. 
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Note: $Z_{n,k}$ crucial with respect to abelian $k$-th powers $\implies \ell_k(n) \leq k^n - 1$.

For $n \geq 4$ and $k \geq 2$, we construct a crucial abelian $k$-power-free word $W_{n,k}$ of length $k^2(n - 1) - k - 1$ using the same method as before.

**Theorem** (G.-Halldórsson-Kitaev, 2008)

For $n \geq 4$ and $k \geq 2$, we have $\ell_k(n) \leq k^2(n - 1) - k - 1$.

Note: $|W_{n,2}| = 4n - 7$ and $|W_{n,3}| = 9n - 13 \implies W_{n,2}$ and $W_{n,3}$ are minimal crucial words over $A_n$ avoiding abelian squares and abelian cubes, respectively.
Upper bound for $\ell_k(n)$

Note: $Z_{n,k}$ crucial with respect to abelian $k$-th powers $\implies \ell_k(n) \leq k^n - 1$.

For $n \geq 4$ and $k \geq 2$, we construct a crucial abelian $k$-power-free word $W_{n,k}$ of length $k^2(n - 1) - k - 1$ using the same method as before.

**Theorem (G.-Halldórsson-Kitaev, 2008)**

For $n \geq 4$ and $k \geq 2$, we have $\ell_k(n) \leq k^2(n - 1) - k - 1$.

Note: $|W_{n,2}| = 4n - 7$ and $|W_{n,3}| = 9n - 13 \implies W_{n,2} \text{ and } W_{n,3}$ are minimal crucial words over $A_n$ avoiding abelian squares and abelian cubes, respectively.

**Conjecture (G.-Halldórsson-Kitaev, 2008)**

For $k \geq 4$ and sufficiently large $n$, the length of a minimal crucial word over $A_n$ avoiding abelian $k$-th powers is given by $k^2(n - 1) - k - 1$. 
Outline

1 Background
   - Repetitions & patterns in words
   - Crucial words & abelian powers

2 Minimal crucial words avoiding abelian cubes
   - Upper bound for length
   - Lower bound for length

3 Minimal crucial words avoiding abelian $k$-th powers
   - Upper bound for length

4 Further research
Problem 1 – Prove or disprove the conjecture: \( \ell_k(n) = k^2(n - 1) - k - 1. \)
Problem 1 – Prove or disprove the conjecture: $\ell_k(n) = k^2(n - 1) - k - 1$.

Problem 2 – Maximal words of minimal length.

- A word $W$ over $A_n$ is *maximal* with respect to a given set of prohibitions if $W$ avoids the prohibitions, but $xW$ and $Wx$ do not avoid the prohibitions for any letter $x$ occurring in $W$. 
Further research

Problem 1 – Prove or disprove the conjecture: $\ell_k(n) = k^2(n - 1) - k - 1$.

Problem 2 – Maximal words of minimal length.

- A word $W$ over $A_n$ is maximal with respect to a given set of prohibitions if $W$ avoids the prohibitions, but $xW$ and $Wx$ do not avoid the prohibitions for any letter $x$ occurring in $W$.

- Example: $323121$ is a maximal abelian square-free word over $\{1, 2, 3\}$ of minimal length.
Problem 1 – Prove or disprove the conjecture: $\ell_k(n) = k^2(n - 1) - k - 1$.

Problem 2 – Maximal words of minimal length.

- A word $W$ over $A_n$ is maximal with respect to a given set of prohibitions if $W$ avoids the prohibitions, but $xW$ and $Wx$ do not avoid the prohibitions for any letter $x$ occurring in $W$.

- Example: 323121 is a maximal abelian square-free word over $\{1, 2, 3\}$ of minimal length.

- The length of a minimal crucial word gives a lower bound for the length of a shortest maximal word.
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Question: Can our approach improve Bullock’s result or can it provide an alternative solution?
Thank you!