Border marriage: matching of contours of serial sections

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Abstract: Surface triangulation methods are often used to reconstruct biomedical objects from serial tomographs. The contours extracted from such images must be matched prior to surface tiling. Previously, researchers used relatively simple objects with trivial matching conditions, or used a priori knowledge, or matched the contours manually. This paper describes a new method for matching contours based on topological considerations. This method can automatically tile complex objects; particular attention is placed on objects with hollow interiors. The method also detects complex hole bifurcations and generates appropriate saddle points. Applied to biomedical modelling, this method can match complex contours with a high degree of accuracy.

1 Introduction

Many researchers have studied reconstructions of biomedical objects from serial sections. Often the objects are characterised by boundary surfaces constructed from a mosaic of primitive patches. The have what are known as a piecewise linear compact orientable 2-manifolds as the boundary. Often the patches are triangles [1-3], although other shapes have been used [4, 5]. A key procedure in the reconstruction process is the tiling algorithm, which takes borders of adjacent sections and produces a set of tiles to represent part of the object's boundary surface. Before tiling can be done, the various borders of adjacent sections must be matched together. Thus, a contour matching method is required, which considers the different borders on each level and attempts to match the contours to the various borders on the next level. This process, called a piecewise linear cobordism, is repeated for all levels of the object. Fig. 1 shows three possible cobordisms between a pair of circles in each of the two sections. One might reasonably prefer the cobordism shown in Fig. 1a on the grounds of simplicity. Frequently, more than one border on one level will, quite correctly, correspond to the same single border on the next level. In the worst case, the contour matching algorithm may content with a many-to-many 'mating', where several contours on one level correspond to many on another level, as in Fig. 1b. Hollow interiors must also be matched and tiled, and this is a recursive problem, because these hollow interiors may also surround additional solid regions, and so on. At the other extreme, a contour may be unmatched to any other contour, and may therefore represent a flat surface to be tiled on its own.

The tiling process has been discussed extensively [6-10]. However, little research has been conducted into the matching process, as previous authors [1, 2] have relied on a priori knowledge or manually entered matching conditions. This is unsatisfactory for an automated 3D reconstruction system. This paper describes a new matching approach that can assign a border matching to sections of an arbitrary solid object. It does not use a priori knowledge about the type of object being modelled. Instead, it uses the fact that the object is a real 3D structure, the shape of which is deduced from the shape of the contours and the topological environment which is provided by the border tracing algorithms [11-13].

2 Definitions

Border tracing of a two dimensional pixel array, which is a section of a solid object, i.e. a tomograph, provides two
kinds of information. First, it gives the border as a set of disjoint loops, each loop being specified as a chain-coded path in the pixel array which returns to its starting point without crossing itself. Secondly, it gives a surroundedness tree [12] with information of a topological nature, specifying which loops are inside which other loops. Loops may be degenerate when they have empty interiors; if they have many pixels they are called ridges, and if a loop has only one pixel it is called a point. We shall refer to the loops in the rim border as borders.

The surroundedness tree may conveniently be described in the language of genealogy [12]. If a surrounds borders \(\beta\) and \(\psi\), then \(\alpha\) is said to be the parent of \(\beta\) and \(\psi\); \(\beta\) and \(\psi\) are the children of \(\alpha\). Borders \(\beta\) and \(\psi\) are called siblings. Borders with no parents are said to be outermost. Outermost borders lie on a surface bounding a solid object; every odd generation is a solid border, i.e. a section of surface bounding a solid, and every even generation is a hole border, i.e. a section of a hole in the object. It is assumed that solid borders are traced clockwise and hole borders are traced counterclockwise.

The sections of a solid object are arranged in a linear ordering, and adjacent sections and hence adjacent surroundedness trees are referred to in this paper. The genealogical terminology is further extended to refer to mating the borders in adjacent sections. Borders that are mated, i.e. assigned to be joined by tiles in some way, are said to be married, and the two borders are said to be spouses. If the marriage is one to one as in Fig. 1a, the marriage is said to be monogamous, otherwise it may be polygamous, or it may be a group marriage as in Fig. 1b. If border \(\alpha\) is married to borders \(\beta\) and \(\psi\), and if border \(\phi\) is also married to \(\psi\), then \(\beta\) and \(\phi\) are indirectly married within the group marriage. This is illustrated in Fig. 1b. If a border cannot be married it is described as celibate.

This terminology will be applied to any pair of adjacent sections; it is started at one end and then iterated to the other end. Thus each border is considered for marriage twice. A border that remains celibate twice is called a disc, and is usually the result of noise or undersampling of the object. When a marriage is not monogamous, the tiling algorithm will have to introduce saddle points, as in Fig. 1b, where two have been introduced. The tiling algorithm must choose the location and height of the saddle points. When a border is celibate, the algorithm has, in effect, to introduce a cup or cap. There are only three possibilities, introducing a cup or cap, introducing a saddle point, or not introducing any more critical points of the ‘height function’. The height (distance to maxima) of cups and caps must be chosen carefully to avoid intersecting cups and caps that may have been introduced into other generations. Similarly, saddle points have to be chosen with care so as not to intersect hole saddle points which may also arise. All the above terminology is illustrated in Fig. 2.

3 Eligibility

A solid object in three-space bounded by a smooth manifold is now considered. Certain things are immediate. First, the boundary must be an orientable compact (smooth) surface, hence each connected component, by well known results in topology, must consist of a sphere with some number of ‘handles’, up to a homeomorphism. The outer boundary may be distinguished from the inner boundary, which may be empty. Secondly, as has been remarked, any planar section of the object yields sections of the boundary which will, in general, be compact orientable 1-manifolds (which is to say contours, which topologically speaking are circles) up to homeomorphism. The solid object, when reconstructed, will have a unique height function which is expected to be a smooth function. It is further assumed that this function will have no degenerate critical points between sections; the justification of this is part of Morse theory [14], and in practice amounts to the observation that an infinitesimal perturbation of a degenerate critical point makes it non-degenerate. The only nondegenerate critical points for surfaces arise from saddle points, maxima and minima. It follows that three operations in constructing the outer and inner bounding surface given the contours in the sections are possible: (i) marry one to one, one to many, or (ii) many to many, or (iii) they may be capped. One to one marriages introduce no new critical points, capping introduces one new critical point per cap, and when \(n\) contours are married to \(m\) contours, \(n + m - 2\) saddle points are introduced. These are the allowable cobordisms. This still yields many possible constructions that are incompatible with the contours being sections of compact orientable surfaces which bound a (not necessarily connected) solid as shown in Fig. 3.

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Fig. 2 Genealogy

A is the parent of B
H and I are children of G
C and J are siblings
H is a point, M is a ridge
H and F are celibate, C is a disc
I and O are monogamous spouses
E, F, H and J are polygamous spouses
A, D, G, C and J are outermost borders
\(i\), \(j\) and \(k\) are saddle points

Fig. 3 Forbidden marriages

A set of constraints on marriages that ensure that the surfaces arising from cobordisms of contours do not need to intersect must, therefore, be developed. If they must intersect, then they clearly cannot comprise part of the boundary of a solid object.
**Proposition 1:** A set of cobordisms which do not intersect comprise the boundary of a solid object contained between the two sections.

**Proof:** For two compact orientable surfaces which do not intersect, two possibilities must be considered: it may be that one is inside the other, or it may not. If one is inside the other, it is clear that the region bounded by both is a connected 3-manifold bounded by the two surfaces. If one is not inside the other, we may 'fill the inside' of each to obtain a disconnected 3-manifold bounded by each surface. The result follows by induction.

An obvious condition to impose is that hole boundaries can only marry solid boundaries and solid boundaries can only marry solid boundaries, but this will not serve the case illustrated in Fig. 4. This is homeomorphic to the standard 'trouser' cobordism between a pair of circles and a single circle, in the case where one of the two circles is inside the other.

Another condition that could be imposed is to only marry outermost borders to other outermost borders. However this is neither necessary nor, in general, desirable, as can be seen by contemplating an axle passing vertically through the middle of a thick washer; there are two marriage patterns which yield solid objects from horizontal sections of the given object; one may obtain either the given object or a solid ball containing a toroidal hole in it.

The following eligibility rules are stipulated:

(i) Outermost borders on one section are eligible to marry outermost borders on an adjacent section.

(ii) Outermost borders may marry other solid borders only if no ancestors of that solid border have married. This operates recursively from the outside in; only after having decided whether ancestors of a border are to marry or not, can the eligibility of a given border be considered.

(iii) (a) A hole border with an unmarried parent is ineligible for marriage. Both it and the parent are capped.

(b) Child borders with a married ancestor are restricted to marrying a subset of borders on the other level. The border's most recently married ancestor (MRMA) is the youngest ancestor in its family tree to have married. The eligibility of the child border is restricted to descendants of the spouses of its MRMA.

(iv) Borders may only marry borders of the same type; i.e solid borders may marry only solid borders and not hole borders, except in the case of mixed marriages which are arranged as in Fig. 4. Here, a solid border and its child hole border may both be married to a solid border by introducing a saddle point. The dual is also possible, with solid and hole borders interchanged. Polygamous mixed marriages are also possible, as shown in Fig. 5.

It can be shown that the restrictions imposed by the eligibility restrictions ensure that the cobordisms never intersect and hence that the result of the marriages arranged in accordance with them yield surfaces bounding a solid object.

**Proposition 2:** Cobordisms constructed subject to the eligibility conditions cannot intersect.

**Proof:** The only way an essential intersection can occur is if crossmarriages occur so that the induced relation between the surroundedness trees does not preserve the partial order of the trees. This possibility is prohibited by applying the rules recursively.

The outmost borders define disjoint regions, and are solid borders. They can be arbitrarily connected to form valid surfaces. Hence rule (i) cannot cause the cobordisms to intersect.

If an outermost border does not marry, it will have no connecting surfaces attached to it, and can be ignored for future eligibility considerations. Similarly, the children of such celibate borders cannot participate in marriage since they are hole borders, and the hole would not have a solid boundary. In fact, a hole border must have a married ancestor before it can marry, which is rule (iii).

The other descendants of the celibate outermost border can be married. Since the grandchildren define regions disjoint from each other and the other outermost borders, they are eligible to marry the outermost borders. This consideration can be applied recursively to the deepest ancestor, and shows that rule (ii) will not result in intersections.

Normally a solid border can be tiled only to other solid borders, and hole borders to other hole borders. If this were not true, the cobordisms would intersect. The exception is a mixed marriage, which treats a married border and an otherwise celibate child as a single border connected by a saddle point. If the child is married, the hole surface would branch at the mixed marriage. If the parent is not married then its children are ineligible for marriage under rule (iii). Thus the bifurcation is valid only if the parent is married but the child is not. This is stated as rule (iv). Rules (i), (ii), (iii) and (iv) define a set of continuous, disjoint, outermost 2-manifolds within the space between the sections. Each of these sections can be considered in turn to find internal holes. The hole regions are limited to the 3D space bounded by the solid surface and top and bottom section planes. The hole regions inside each space will be formed from descendants of the marriage that defines each particular region. Other hole
borders are ineligible since the hole surfaces connecting such external borders would pass through the solid surface. The same restriction binds further solid regions within each hole region. Thus the descendants of a marriage can only marry descendants on the other side of the marriage. Or, from the point of view of the descendant, it may only marry a descendant of a spouse of its MRMA, as stated in the first part of rule (iii)b. This means that the children of the region’s marriage can be considered to be outermost borders (they cannot marry any borders higher in the topology) and the same rules as for outermost solid surfaces can be used to define the interior surfaces. This validates the second part of rule (iii)b.

Consequently, if the four eligibility rules are followed, the border marriages cannot form invalid surfaces. On using these rules, sets of eligible borders can be formed for possible marriages. Within these sets it is necessary to select suitable marriages for tiling. Some criteria are required to select the suitable marriages from the eligible sets. The above arguments using smooth manifolds and smooth cobordisms are applied to piecewise linear manifolds and cobordisms; clearly the main topological features are preserved in this process.

4 Selecting marriages

The simplest cobordism interpolating two sets of borders needs to be specified. There are two criteria for simplicity which seem natural. The first is to introduce the maximum number of new critical points of the height function. Thus Figs. 1a and c introduce no new critical points, but Fig. 1b introduces two. The second criterion is that the tilings should have as little displacement in the xy-plane as possible. Thus Fig. 1a is preferred to Fig. 1c. These two criteria may be in conflict, as indicated in Fig. 6.

Fig. 6 Conflicting criteria

Fig. 6a introduces two new critical points, and Fig. 6b introduces none, but Fig. 6b introduces very severe horizontal displacements. Although there is room for debate about which is 'correct', reflecting the different kinds of objects from which sections might have been taken, Fig. 6a is taken as simpler or more plausible than Fig. 6b.

The first proposed criterion is to marry those borders surrounding regions above the regions of eligible borders. In determining whether borders lie over another, the spaces occupied by descendants are not considered. This can be expressed in set terminology by projecting the child completely overlaps its step-parents, it is considered to be an interior hole, and there is no bifurcation. However, if any part of the hole does not overlap its step-parents, the hole may participate in a mixed marriage, as shown in Fig. 8. Let \( x \) be the child.

Fig. 7 Marriage criteria

A, B and C are on one level, and D, E, F and G are on another level

The second criterion for borders eligible for mixed marriage is based on whether the celibate child is entirely underneath or above its parent’s spouse (step-parents). If the projection of the child completely overlaps its step-parents, it is considered to be an interior hole, and there is no bifurcation. However, if any part of the hole does not overlap its step-parents, the hole may participate in a mixed marriage, as shown in Fig. 8. Let \( x \) be the child.

Fig. 8 Bifurcation criteria

\( a \) Bifurcation, \( b \) Internal hole

region, and \( \beta_{1..n} \) be the set of step-parent regions. Then

\[
\alpha \leq \sum_{i=1}^{n} \beta_i
\]

(2)

then \( x \) is an interior hole. Otherwise it is a bifurcation, and a saddle point is located halfway along some line connecting the child and the parent.

5 Implementation

This method has been programmed in C and occupies about 300 lines of source code. All sections \( S_1, \ldots, S_n \) are matched in sliding pairs. Two empty sections, \( S_0 \) and \( S_{n+1} \), are used to force the topmost and bottom-most borders to be celibate. As each pair is matched, two sets of eligible borders, \( L \) and \( U \), are generated which contain borders from the lower and upper sections respectively. The marriage criteria described below are then applied to determine which borders should be married. Any borders that are married are then removed from \( L \) and \( U \), and the grandchildren of the celibate borders are added. Then the sets are again checked for marriage, although pairs that have already been tested are marked to avoid unnecessary computation. Again, the married borders are removed, and grandchildren of celibate borders are added, and so on, until there are no new borders. This results in all outermost borders being married.

Next, the marriages are considered in turn. For each marriage, let \( L \) and \( U \) contain the children of either side.

of the marriage, on the lower and upper sections. Again, the marriage test is repetitively applied as described above, until no more grandchildren are added. The marriages resulting from this test are similarly checked in turn, to find any marriages amongst children, until no further marriages exist. After all marriages have been formed, any celibate borders with married parents are checked for bifurcation (as described below), and if necessary are inserted into the marriage. After this the pair of sections is fully matched and the next pair can be considered. This continues for all section pairs until the entire border is matched.

6 Fast marriage test

Borders may have up to several hundred vertices, and will be tested for intersection against many other borders. Testing for intersection between such polygons can be performed algebraically, but such methods are extremely slow. Instead, a fast graphical method can be used. A polygon from set $L$ is drawn and its interior is flood-filled with 1s onto an empty frame buffer, and the children are similarly area-filled with 0s. Each of the $U$ polygons is tested by checking all vertices to determine the value of the pixel in the frame buffer. If a 1-pixel is found, then the borders overlap, but if only 0-pixels are found, the borders do not overlap. The frame buffer is cleared, and the process is repeated for the next polygon in $L$. To ensure correct testing, the $U$ borders are then drawn and tested against the $L$ borders. It should be noted that the results of this method could differ from the results of an algebraic method for marginally connected borders, but such differences are insignificant.

If there are $n$ borders in $L$, and $m$ borders in $U$, then the result of this matching can be expressed as an $m \times n$ array where a 1 indicates a match between the particular borders. This is illustrated by the (contrived) example shown in Fig. 9. Interpreting this array is complicated by the indirect marriages, i.e. $a$ to $D$ through $c$ and $E$. For this reason, marriages must be found by scanning the array. When a matched pair is found, its borders are noted in a new set, the reference is zeroed and its row and column are scanned, and so on until no further rows or columns are found. This continues for all entries in the array until the array is empty and all marriage sets have been found. The marriage sets contain enough information for the tiling algorithm to tile the marriages.

7 Fast test for mixed marriages

A border eligible for a mixed marriage can also be tested quickly if a graphical rather than an algebraic test is used. The child's step-parents are area filled with 1s on an empty frame buffer. The child's vertices are tested, and if a 0-pixel is found in the frame buffer then the child is a bifurcation to be inserted into its parent's marriage.

Locating a suitable saddle point is relatively slow. The saddle point lies on a line segment connecting a child vertex with a 0-pixel in the frame buffer, and any parent vertex. All possible such segments are considered in turn. If the line intersects the border of a step-parent, it is rejected. Otherwise the perpendicular distance $D$ to the nearest step-parent vertex is found, and the length $L$ of the line segment is computed. The line segment with the smallest value of $(L^2 - D^2)$ is chosen as the best line segment between the child and its parent. The saddle point is located on the line's midpoint, and the height of the saddle point is computed using the same formula for saddle heights used by the tiling algorithm. In a very complex object, all line segments could conceivably be rejected, as the saddle ridge is not a straight line. Current software cannot handle this and must cancel the mixed marriage.

8 Results

This new algorithm has been tested on a wide range of images. It has proven to be a very robust method for tiling artificial objects and generates correct tiling commands for complex objects such as interlocking rings. It has also performed well when tested on real, biomedical images. For example, Fig. 10 shows a vertebra that has been reconstructed from CT images. It cannot match grossly undersampled images, which can result in borders appearing above extraneous eligible partners, or moving away from true matches. Further work on the marriage criteria could improve this. Further work could also be directed towards a better method for locating the mixed marriage saddle points. This is currently slow, and sometimes the saddle point is poorly located. However, generally, the algorithm works quickly and effectively.

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