On a generalisation of trapezoidal words

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Joint work with Florence Levé (Université de Picardie – Jules Verne).

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Words

By a \textit{word}, I mean a \textbf{finite or infinite sequence} of symbols (\textit{letters}) taken from a non-empty finite set \( A \) (\textit{alphabet}).

\textbf{Examples:}

\begin{itemize}
\item 001
\item \((001)^\infty = 00100100100100100100100100100100100 \ldots \)
\item 1100111100011011101111001101110010111111101 \ldots
\item 100102110122220102110021111102212222201112012 \ldots
\item 1121212121212 \ldots
\end{itemize}
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- $[1; 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, \ldots] = \sqrt{3}$
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**Basic measure:** number of distinct blocks (factors) of each length occurring in the word.
Words: Factor Complexity

- Given a finite or infinite word $w$, let $F_n(w)$ denote the set of distinct factors of $w$ of length $n \in \mathbb{N}^+$. 
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Conjecture: \( C_x(n) = 2^n \) for all \( n \) as it is believed \( \sqrt{2} \) is normal in base 2.
Complexity & Periodicity

**Theorem** (Morse-Hedlund 1940)

An infinite word $w$ is **eventually periodic** if and only if $C_w(n) \leq n$ for some $n \in \mathbb{N}^+$. 
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An infinite word $\mathbf{w}$ is **eventually periodic** if and only if $C_w(n) \leq n$ for some $n \in \mathbb{N}^+$. That is: $\mathbf{w}$ is aperiodic $\iff C_w(n) \geq n + 1$ for all $n \in \mathbb{N}$. 

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- Numerous equivalent definitions & characterisations . . .
A Characterisation by Palindromic Complexity

Given a finite or infinite word $w$, let $P_w(n)$ denote the *palindromic complexity function* of $w$, which counts the number of palindromic factors of $w$ of each length $n \geq 0$. 
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**Theorem** (Droubay-Pirillo 1999)

An infinite word $w$ is Sturmian if and only if

$$P_w(n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd} \end{cases}$$
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- A Sturmian word over the alphabet $\{a, b\}$ contains either $aa$ or $bb$, but not both.
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What do such words look like? And how can we construct them?
Constructing Sturmian words

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Consider a line (call it $\ell$) of the form:

$$y = \alpha x + \rho$$

where $\alpha$ is an irrational number in $(0, 1)$ and $\rho \geq 0$. 
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- Let \( \mathcal{P} \) denote the path along the integer lattice that starts at the point \((1, 0)\) below the line \( \ell \) with the property that the region in the plane enclosed by \( \mathcal{P} \) and \( \ell \) contains no other points in \( \mathbb{Z} \times \mathbb{Z} \) besides those of the path \( \mathcal{P} \).
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- The so-called Sturmian word of slope \( \alpha \) and intercept \( \rho \) is obtained by coding the steps of the path \( \mathcal{P} \).
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The so-called Sturmian word of slope \( \alpha \) and intercept \( \rho \) is obtained by coding the steps of the path \( \mathcal{P} \).

– A horizontal step is denoted by the letter \( a \).
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- Let’s consider a nice geometric realisation of infinite Sturmian words . . .
- Consider a line (call it $\ell$) of the form:

$$y = \alpha x + \rho$$

where $\alpha$ is an irrational number in $(0, 1)$ and $\rho \geq 0$.

- Let $P$ denote the path along the integer lattice that starts at the point $(1, 0)$ below the line $\ell$ with the property that the region in the plane enclosed by $P$ and $\ell$ contains no other points in $\mathbb{Z} \times \mathbb{Z}$ besides those of the path $P$.

- The so-called Sturmian word of slope $\alpha$ and intercept $\rho$ is obtained by coding the steps of the path $P$.
  - A horizontal step is denoted by the letter $a$.
  - A vertical step is denoted by the letter $b$. 
Sturmian words: Construction by example

\[ y = \frac{\sqrt{5} - 1}{2} x \quad \rightarrow \quad \text{Fibonacci word} \quad (\text{Standard Sturmian word of slope } \frac{\sqrt{5} - 1}{2}) \]
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\[ f = \text{abaababaab} \ldots \]
Fibonacci word: \( f = \text{abaababaabaabaabaaba} \cdots \)

- The Fibonacci numbers show up in connection with many combinatorial properties of the Fibonacci word \( f \).
**Fibonacci word**: $f = abaababaababaababaababaabaabaabaaba\cdots$

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- For instance, the Fibonacci word begins with arbitrarily long palindromes, starting with
  
  $\varepsilon$ (empty word), $a$, $aba$, $abaaba$, $abaababaaba$, $abaababaabaaba$, $\ldots$
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  \( \varepsilon \) (empty word), \( a \), \( aba \), \( ababa \), \( abababaabaabaabaabaabaabaabaabaabaabaabaabaaba \), \( \ldots \)

And it can be shown that the palindromic prefixes of \( f \) have lengths

\[
\{F_{n+1} - 2\}_{n \geq 1} = 0, 1, 3, 6, 11, 19, \ldots
\]

where \( \{F_n\}_{n \geq 0} \) is the sequence of Fibonacci numbers

\( 1, 1, 2, 3, 5, 8, 13, 21, \ldots \), defined by: \( F_0 = F_1 = 1, F_n = F_{n-1} + F_{n-2} \)

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Standard Sturmian words

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  In fact, such words have a purely combinatorial construction using the iterated palindromic closure operator . . .
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- The iterated palindromic closure operator (Justin, 2005) is denoted by $\text{Pal}$ and is defined as follows.
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Standard Sturmian words: Palindromic Construction

**Theorem** (de Luca 1997)

An infinite word $s$ over $\{a, b\}$ is a **standard Sturmian word** if and only if there exists an infinite word $\Delta = x_1 x_2 x_3 \cdots$ over $\{a, b\}$ (not of the form $ua^\infty$ or $ub^\infty$) such that

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- **Example**: Fibonacci word is directed by $\Delta = (ab)(ab)(ab)\cdots$
Recall: Fibonacci word

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Example

Graph of the factor complexity of the finite Sturmian word $aabaabab$
Trapezoidal Words . . .

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We say that finite word $w$ with alphabet $\mathcal{A}$ (of size $|\mathcal{A}| \geq 2$) is a generalised trapezoidal word (or GT-word for short) if the graph of its factor complexity $C_w(n)$ as a function of $n$ (for $0 \leq n \leq |w|$) is either constant or a regular trapezoid (possibly an isosceles triangle) on the interval $[1, |w| - |\mathcal{A}| + 1]$. 
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![Diagram of Generalised Trapezoidal Words](image-url)
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Clearly these words coincide with the (original) trapezoidal words when $|\mathcal{A}| = 2$. 
Some Examples

Length 10 over $\mathcal{A} = \{a, b, c\}$

<table>
<thead>
<tr>
<th>GT-word</th>
<th>$C(n)$ for $n = 0, 1, 2, \ldots, 10$</th>
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<tbody>
<tr>
<td>aaaaaaaaabc</td>
<td>1, 3, 3, 3, 3, 3, 3, 3, 3, 2, 1</td>
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Length 8 over $\mathcal{A} = \{a, b, c, d\}$

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If $w$ is a GT-word, then each factor of $w$ (containing at least two different letters) is also a GT-word.

Moreover, the language of all GT-words is closed under reversal.

**Theorem (G.-Levé 2011)**

A finite word $w$ is a GT-word if and only if its reversal is a GT-word.
Binary Case

In the case when $|\mathcal{A}| = 2$, we have proved the following.

**Theorem (de Luca-G.-Zamboni 2008)**

Let $w$ be a binary palindrome. Then $w$ is trapezoidal if and only if $w$ is Sturmian.
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That is, trapezoidal words (and hence finite Sturmian words) are “rich” in palindromes in the sense that they contain the maximum number of distinct palindromic factors, according to the following result.
Trapezoidal Words
A Generalisation

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**Theorem** (Droubay-Justin-Pirillo 2001)

A finite word \(w\) contains at most \(|w| + 1\) distinct palindromes (including \(\varepsilon\)).
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Definition (G.-Justin 2007)

A finite word $w$ is rich iff $w$ contains exactly $|w| + 1$ distinct palindromes.
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- Any binary trapezoidal word is rich, but not conversely.
  
  E.g., $aabbaa$ is rich, but not trapezoidal ($C(1) = 2$, $C(2) = 4$)

Roughly speaking, a finite or infinite word is rich if and only if a new palindrome is introduced at each new position.

Example: $abaabaaabaaaabaaaaaabab \cdots$
Rich Words

**Definition (G.-Justin 2007)**

A finite word $w$ is *rich* iff $w$ contains exactly $|w| + 1$ distinct palindromes.

**Examples:**

- $abac$ is rich, whereas $abca$ is not rich.
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Richness & GT-words when $|\mathcal{A}| \geq 3$

Unlike in the binary case ($|\mathcal{A}| = 2$), **not** all GT-words are palindromic-rich.
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**Example**

The GT-word $ababadbc$ is not rich since it contains a non-palindromic complete return to $b$, namely $badb$. 
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However, all palindromic GT-words are rich by the following more general result.
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The GT-word $ababadbc$ is not rich since it contains a non-palindromic complete return to $b$, namely $badb$.

However, all palindromic GT-words are rich by the following more general result.

Theorem

Suppose $w$ is a GT-word and let $v$ denote the unique factor of $w$ such that $w = bve$ where $b$ is the longest (possibly empty) prefix of $w$ such that $|w|_x = 1$ for each $x \in \text{Alph}(b)$ and $e$ is the longest (possibly empty) suffix of $w$ such that $|w|_x = 1$ for each $x \in \text{Alph}(e)$.

If $v$ is a palindrome, then $w$ is rich.
Examples

- The GT-word \( w = abacabade \) has \( v = abacaba \) (a palindrome) and \( w \) is indeed rich.
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- The GT-word $w = abacabade$ has $v = abacaba$ (a palindrome) and $w$ is indeed rich.

- The converse of the theorem does not hold.
Examples

- The GT-word \( w = abacabade \) has \( v = abacaba \) (a palindrome) and \( w \) is indeed rich.

- The converse of the theorem does not hold. For example, the GT-word \( ababadae \) is rich, but the corresponding \( v \) is \( ababada \) (non-palindromic).
Thank You!

Dammit, I’m mad!

* Both phrases are (rich) palindromes! *